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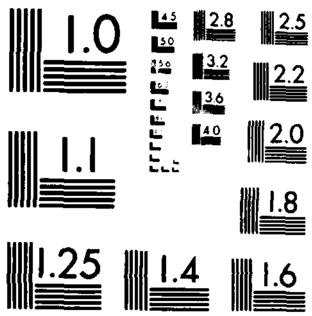

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THERMOMECHANICAL CRACKING IN LAYERED MEDIA
FROM MOVING FRICTION LOAD

BY

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ABSTRACT

This investigation studies a two-dimensional model of a layered medium subjected to the frictional excitation of a high-speed asperity traversing over the surface of the medium. The general analytical solutions of the mechanical stress state, the temperature field and the thermal stress state are obtained and expressed in the Fourier transform expressions. Numerical solutions are carried out for cases of uniform and parabolic pressure distributions. The resulting stress state yields the conditions for the asperity to initiate cracks in the layered medium. This paper studies, further, the thickness effect of the surface layer, the effect of relative stiffness of the surface layer and the substrate, and the effect of an insulating layer versus a conductive layer.

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Nomenclature

c	specific heat
c_1, c_1^*	The dilatational wave speed of the surface layer and the substrate, respectively.
c_2, c_2^*	The speed of shear wave of the surface layer and the substrate, respectively
H	thickness of the surface layer
k_1, k_2	thermal conductivity
ℓ	asperity characteristic dimension, the half width of the contact area
M	Mach number
P	pressure over the contact area
P_o	average pressure over the contact area
q	heat flux through the contact area
q_o	average heat flux through the contact area
R_1, R_2	Peclet numbers of the surface layer and the substrate, respectively
T	temperature
u_1, u_2	displacement in x_1 and x_2 direction, respectively
U, V	dimensionless displacement in x_1 and x_2 direction, respectively
v	traverse speed of asperity
α	coefficient of thermal expansion
δ_{ij}	Kronecker delta
$\epsilon_{11}, \epsilon_{12}, \epsilon_{22}$	strain field
$\{\xi, \eta\}$	dimensionless coordinates (= x_i/ℓ)
κ	thermal diffusivity
λ, μ	Lamé coefficient
μ_f	Coulomb coefficient of friction

$\sigma_{11},\sigma_{12},\sigma_{22}$	stress field
$\sigma_{\xi\xi},\sigma_{\xi\eta},\sigma_{\eta\eta}$	dimensionless stress field
ρ	mass density

CHAPTER I

INTRODUCTION

1.1 Purpose of the Investigation

This investigation addresses the general behavior of a failure mechanism that is caused by a high speed asperity traversing over a layered surface as shown in Figure 1. The resulting cracks on the surface lead eventually to total failure of the devices. Such a phenomenon of failure has been observed in many a pair of mating surfaces rubbing against each other such as brakes, marine seals, and the like.

The nominal design pressure between such devices is based upon the total mating surface. However, at the operating speed, the actual contact area can be smaller by a factor of the order of 10^{-3}, or even 10^{-4}. Such reduced contact area could be the result of an asperity or the cause thereof. As a result, a low design pressure may result in a very high interfacial pressure. This causes a large frictional force in the contact area. The high friction would cause locally an extremely high temperature, leading to cracking of the surface. In view of the frequency of such phenomenon, many attempts of surface modification have been under investigation. Such current significant development underlies the importance of the present investigation in the thermo-mechanical behavior of layered surface under the frictional excitation of a high speed traversing asperity.

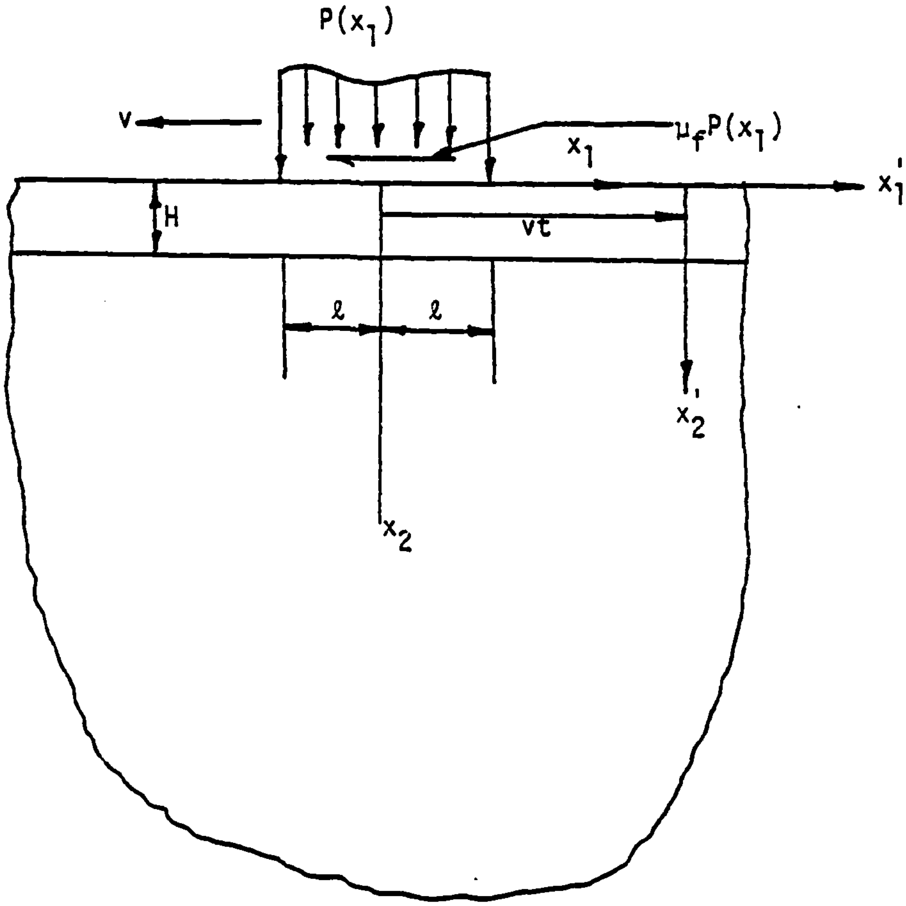

Figure 1. Two-dimensional Asperity.

1.2 General Background and Related Investigation in Progress

Two bodies in sliding contact under heavy loads will experience high local temperatures near the contacting surfaces because of excessive frictional heating. Cracking may then occur in the neighborhood of the contact zone due to the combination of thermal heating and mechanical load. This phenomenon is known as heat checking [1].

One of the models of failure is due to high pressure, and dry friction exists as a result of asperities which are developed on the mating surfaces. The asperity may be either an external or internal material inclusion or some thermomechanical deformation of the mating surfaces. The asperity represents a moving concentration of pressure and excess Coulomb friction on the mating surfaces. Friction influences both the mechanical stress state and the thermal stress state because frictional heating produces a severe thermal stress state. These high stresses can then initiate a fracture of the material.

A general survey of the problem of cracking through the development of a frictional hot spot was discussed by Burton [2]. The existence of high hot spots was demonstrated experimentally by Sibley and Allen [3], who showed systematically moving hot patches in the contact zone. Two and three-dimensional models of heat checking in the contact zone of a bearing seal were presented by Ju, et al. [4].

In both the two-dimensional and three-dimensional models of a single moving asperity, the high temperature field is concentrated in a thin layer in the neighborhood of the surface. The depth of such a surface layer is of the order of twenty percent of the asperity size. For a material such as stellite III, the high temperature field will

not alter significantly the important mechanial properties. Such may not be the case for other materials. When property changes occur, we will have a problem of friction cracking involving a solid with a surface layer of different material properties. Problems of such phenomena and those of coated surfaces belong to the general class of thermocmechanical cracking of layered media. The present investigation deals then with the problem of heat checking in media with a hard surface layer. The surface layer may be softer or harder than the substrate. Their thermal properties may be different. These factors and the thickness of the surface layer constitute the parameters of analysis in the investigation.

The analytical model in the investigation is based on the genei theory of a continuum. From observation of failed specimens near the surface, it is reasonable to postulate that for hard surfaces the plastic deformation and rupture at the surface are at the granular or even the subgranular level. The base solid material subjected to the asperity friction is essentially elastic. The irrecoverable work in the surface deformation manifests as heat input.

The surface layer, be it thermally induced or coated, is assumed to be of uniform thickness. The depth of the substrate is infinite in comparison to the asperity size.

1.3 General Theory

1.3.1 The uncoupled theory of thermoelasticity [5]

The basic mathematical formulation of thermoelasticity describing the behavior of continuous media are the following equations:

$$kT_{,ii} = \rho c_v \dot{T} + (3\lambda + 2\mu)\ \alpha T_o\ \dot{\varepsilon}_{kk} \quad (1)$$

$$\sigma_{ij,j} = \rho \ddot{u}_i \quad (2)$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (3)$$

$$\sigma_{ij} = \delta_{ij}\lambda\varepsilon_{kk} + 2\mu\varepsilon_{ij} - \delta_{ij}\ (3\lambda + 2\mu)\alpha(T - T_o) \quad (4)$$

The coupled dynamic formulation can be simplified for cases without thermal or mechanical shocks to the uncoupled quasi-static formulation.

When an external mechanical agency produces variations of strain within a body, the time rate of strain variations cause a change in the temperature as indicated by the heat conduction equation. Consequently, by a flow of heat, the whole process increases entropy and therefore increases the energy stored in a mechanically irrecoverable manner. This phenomenon, known as thermoelastic dissipation, is accounted for by the mechanical term; therefore omitting it may yield erroneous results. However, the dilatation rate in equation (1) is by order of magnitude small for most of the thermoelastic problems. The coupled heat equation, [equation (1)] may be rewritten as

$$kT_{,ii} = \rho c_v \dot{T}\left[1 + \delta\left(\frac{\lambda + 2\mu}{3\lambda + 2\mu}\right)\left(\frac{\dot{\varepsilon}_{kk}}{\alpha\dot{T}}\right)\right] \quad (5)$$

where the nondimensional parameter δ is defined by

$$\delta = \frac{(3\lambda + 2\mu)^2\alpha^2 T_o}{\rho^2\ c_v\ Ve^2} \quad (6)$$

and Ve is the dilatational wave speed.

The coupling term is negligible compared to unity if

$$\frac{\dot{\epsilon}_{kk}}{3\alpha\dot{T}} << \left(\frac{\lambda + 2\mu/3}{\lambda + 2\mu}\right)\frac{1}{\delta} \tag{7}$$

For illustration, values of the parameter δ will be computed for purposes of numerical comparisons. For aluminum, with

$\lambda = 4.13 \times 10^{10}$ pa, $\mu = 2.76 \times 10^{10}$ pa,

$\alpha = 2.3 \times 10^{-5}/°C$, $\rho = 2.7 \times 10^{3}$ kgs/m^3

$c_v = 835$ N-m/(kg °C),

and, taking as an example $T_0 = 90°C$, we obtain $\delta = 0.029$.

For steel, with

$\lambda = 12.4 \times 10^{10}$ pa, $\mu = 8.3 \times 10^{10}$ pa, $\alpha = 1.2 \times 10^{-5}/°C$,

$\rho = 7.8 \times 10^{3}$ kgs/m^3, $C_V = 459$ N-m/(kg °C), and again with

$T_0 = 90°C$, the corresponding value is $\delta = 0.014$.

Thus for both cases the coupling will be small [from equation (7)] if approximately,

$$\frac{\dot{\epsilon}_{kk}}{3\alpha\dot{T}} << 20.$$

For temperature distributions with no sharp variations or discontinuities in their time histories, the time rate change of the dilatation is of the same order of magnitude as that of the temperature;

thus disregarding the coupling term as described previously is reasonable.

The preceding discussion makes it clear that the possibility of omitting the coupling terms depends not only on the fact that the inequality $\delta \ll 1$ must hold (as it does for most metals), but also on the fact that strain rates must be at most of the same order of magnitude as temperature rates. The latter condition implies that the time history of the displacements closely follows that of the temperature; in other words no pronounced lag or vibrations in the motion of the body must arise.

1.3.2 Brittle Fracture [6]

When a solid is subjected to increasing loads, the resulting stresses will, at a certain stage, become high enough to cause the solid to break apart. If such breakage comes about before the piece has thinned down to zero thickness, it is called fracture, and if the amount of permanent deformation preceding fracture is negligible, it is called brittle fracture. In this investigation, both the surface layer and the substrate materials are brittle so that when the stress reaches the ultimate stress, brittle fracture will occur. Also, if the shear stress reaches the maximum shear stress at the interface, shear delamination will happen.

CHAPTER II

MATHEMATICAL MODEL

The problem under consideration is one of thermomechanical cracking in a layered medium by a fast moving asperity, as shown in Figure 1. Because the actual contact area of the moving asperity is much smaller than the layered medium, the mathematical model is represented by a half-space subjected to a fast moving asperity whose effect is delineated into a moving heat source and a moving mechanical load of combined pressure and tangential friction. Two sets of coordinates are employed: $x_1' - x_2'$ are fixed to the medium, $x_1 - x_2$ are fixed to the moving load. The relative speed of the contact surface is assumed to be large enough to result in a high Péclet number ($v\ell/\kappa$) but smaller than the Rayleigh wave speed. The quasi-static theory thus holds. The general solutions for the mechanical stress state, the temperature field and the thermal stress state are approached by the use of the Fourier transform method. Because of their complexity, the general solutions are left in the transformed space. Solutions by numerical integrations of the inverse transforms are carried out for special problems.

2.1 Basic equations

The governing equations come from Cauchy's law and the uncoupled theory of thermoelasticity, in terms of the moving convective coordinates $\{x_i\}$. The acceleration in Cauchy's law will have only the convective terms. Hence, Navier's equation is expressed as

$$(\lambda + \mu)\, u_{k,ki} + \mu u_{i,kk} - (3\lambda + 2\mu)\, \alpha\, T_{,i} = \rho v^2 \frac{\partial^2 u_i}{\partial x_1^2}$$

where summation convention is used for repeated indices of roman minuscules.

For i = 1

$$(\lambda + 2\mu - \rho v^2) \frac{\partial^2 u_1}{\partial x_1^2} + (\lambda + \mu) \frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \mu \frac{\partial^2 u_1}{\partial x_2^2} - (3\lambda + 2\mu)\, \alpha \frac{\partial T}{\partial x_1} = 0 \;.$$

For i = 2

$$(\mu - \rho v^2) \frac{\partial^2 u_2}{\partial x_1^2} + (\lambda + \mu) \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + (\lambda + 2\mu) \frac{\partial^2 u_2}{\partial x_2^2} - (3\lambda + 2\mu)\, \alpha \frac{\partial T}{\partial x_2} = 0$$

The Hooke's law equations are

$$\sigma_{11} = (\lambda + 2\mu) \frac{\partial u_1}{\partial x_1} + \lambda \frac{\partial u_2}{\partial x_2} - (3\lambda + 2\mu)\, \alpha(T - T_o) \;,$$

$$\sigma_{12} = \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right),$$

$$\sigma_{22} = \lambda \frac{\partial u_1}{\partial x_1} + (\lambda + 2\mu) \frac{\partial u_2}{\partial x_2} - (3\lambda + 2\mu)\, \alpha(T - T_o) \;.$$

The above equations apply to both the surface layer and the substrate. The surface boundary is traction prescribed so that

$$\sigma_{12}^{(1)} = \mu_f\, p(x_1) \qquad (|x_1| \le \ell \;,\; x_2 = 0) \;,$$

$$\sigma_{22}^{(1)} = \begin{cases} 0 & (|x_1| > \ell \ , \ x_2 = 0) \\ -p(x_1) & (|x_1| \leq \ell \ , \ x_2 = 0) \end{cases} .$$

Regularity conditions hold at infinity.

$$\sigma_{11}, \sigma_{12}, \sigma_{22}, u_1, u_2 \to 0 \qquad (0 \leq x_2 \leq H \ , \ |x_1| \to \infty)$$

$$\sigma_{11}, \sigma_{12}, \sigma_{22}, u_1, u_2 \to 0 \qquad (x_1^2 + x_2^2 \to \infty)$$

Continuity conditions apply at the surface layer/substrate interface

$$\sigma_{12}^{(1)} = \sigma_{12}^{(2)} , \ \sigma_{22}^{(1)} = \sigma_{22}^{(2)} , \ u_1^{(1)} = u_1^{(2)} , \ u_2^{(1)} = u_2^{(2)} \qquad (x_2 = H)$$

where the superscript (1) designates the surface layer, (2) the substrate.

The heat conduction equations are

$$\frac{\partial^2 T^{(1)}}{\partial x_1^2} + \frac{\partial^2 T^{(1)}}{\partial x_2^2} = \frac{V}{\kappa_1} \frac{\partial T^{(1)}}{\partial x_1} ,$$

$$\frac{\partial^2 T^{(2)}}{\partial x_1^2} + \frac{\partial^2 T^{(2)}}{\partial x_2^2} = \frac{V}{\kappa_2} \frac{\partial T^{(2)}}{\partial x_1} .$$

The boundary conditions, the continuity conditions, and the regularity conditions for the heat equations are

$$-k_1 \frac{\partial T^{(1)}}{\partial x_2} = \begin{cases} q(x_1) & (|x_1| \leq \ell \ , \ x_2 = 0) \\ 0 & (|x_1| > \ell \ , \ x_2 = 0) \end{cases}$$

$$k_1 \frac{\partial T^{(1)}}{\partial x_2} = k_2 \frac{\partial T^{(2)}}{\partial x_2} \qquad (x_2 = H)$$

$$T^{(1)} = T^{(2)} \qquad (x_2 = H)$$

$$T^{(1)} \, , \, T^{(2)} \to 0 \qquad (x_1^2 + x_2^2 \to \infty) \, .$$

The following dimensionless quantities will be used for general computation.

$$\xi = x_1/\ell \, , \; \eta = x_2/\ell \, , \; D = H/\ell \, , \; \sigma_{\xi\xi} = \sigma_{11}/P_o \, , \; \sigma_{\eta\eta} = \sigma_{22}/P_o \, ,$$

$$\sigma_{\xi\eta} = \sigma_{12}/P_o \, , \; u = u_1/\ell \, , \; V = u_2/\ell \, , \; M = v/c_2 \, , \; N = c_1/c_2 \, ,$$

$$c_1 = \left(\frac{\lambda_2 + 2\mu_2}{\rho_2}\right)^{1/2} , \; c_2 = \left(\frac{\mu_2}{\rho_2}\right)^{1/2} , \; c_1^* = \left(\frac{\lambda_1 + 2\mu_1}{\rho_1}\right)^{1/2} , \; c_2^* = \left(\frac{\mu_1}{\rho_1}\right)^{1/2} ,$$

$$b = \left(\frac{3\lambda_2 + 2\mu_2}{\rho_2}\right)^{1/2} , \; b^* = \left(\frac{3\lambda_1 + 2\mu_1}{\rho_1}\right)^{1/2} , \; \gamma_1 = \frac{q_o \ell \alpha_1}{k_1} ,$$

$$\gamma_2 = \frac{q_o \ell \alpha_2}{k_1} , \; I = \frac{c_1^*}{c_2} , \; J = \frac{c_2^*}{c_2} , \; \delta = \frac{\rho_1}{\rho_2} , \; P' = \frac{P}{P_o} ,$$

$$\phi^{(i)} = (T^{(i)} - T_o) \, k_1/q_o\ell \, , \; Q = \frac{q}{q_o} \, , \; R_1 = v\ell/\kappa_1 \, , \; \text{and} \; R_2 = v\ell/\kappa_2 \, .$$

The linear theory of elasticity allows the equation to be delineated into the mechanical and the thermal parts. They are thus grouped into (1) equations for the mechanical stress field, (2) equations for the temperature field, and (3) equations for the thermal stress field.

2.1.1 Mechanical stress field

In the surface layer

$$(I^2 - M^2) \frac{\partial^2 u^{(1)}}{\partial \xi^2} + (I^2 - J^2) \frac{\partial^2 v^{(1)}}{\partial \xi \partial \eta} + J^2 \frac{\partial^2 u^{(1)}}{\partial \eta^2} = 0 , \tag{8}$$

$$(J^2 - M^2) \frac{\partial^2 v^{(1)}}{\partial \xi^2} + (I^2 - J^2) \frac{\partial^2 u^{(1)}}{\partial \xi \partial \eta} + I^2 \frac{\partial^2 v^{(1)}}{\partial \eta^2} = 0 , \tag{9}$$

$$\sigma_{\xi\xi}^{(1)} = \delta \frac{\mu_2}{P_o} \left[I^2 \frac{\partial^2 u^{(1)}}{\partial \xi^2} + (I^2 - 2J^2) \frac{\partial v^{(1)}}{\partial \eta} \right] , \tag{10}$$

$$\sigma_{\xi\eta}^{(1)} = \delta J^2 \frac{\mu_2}{P_o} \left[\frac{\partial u^{(1)}}{\partial \eta} + \frac{\partial v^{(1)}}{\partial \xi} \right] , \tag{11}$$

$$\sigma_{\eta\eta}^{(1)} = \delta \frac{\mu_2}{P_o} \left[(I^2 - 2J^2) \frac{\partial u^{(1)}}{\partial \xi} + I^2 \frac{\partial v^{(1)}}{\partial \eta} \right] . \tag{12}$$

In the substrate region

$$(N^2 - M^2) \frac{\partial^2 u^{(2)}}{\partial \xi^2} + (N^2 - 1) \frac{\partial^2 v^{(2)}}{\partial \xi \partial \eta} + \frac{\partial^2 u^{(2)}}{\partial \eta^2} = 0 , \tag{13}$$

$$(1 - M^2) \frac{\partial^2 v^{(2)}}{\partial \xi^2} + (N^2 - 1) \frac{\partial^2 u^{(2)}}{\partial \xi \partial \eta} + N^2 \frac{\partial^2 v^{(2)}}{\partial \eta^2} = 0 , \tag{14}$$

$$\sigma_{\xi\xi}^{(2)} = \frac{\mu_2}{P_o} \left[N^2 \frac{\partial u^{(2)}}{\partial \xi} + (N^2 - 2) \frac{\partial v^{(2)}}{\partial \eta} \right] , \tag{15}$$

$$\sigma_{\xi\eta}^{(2)} = \frac{\mu_2}{P_o} \left[\frac{\partial u^{(2)}}{\partial \eta} + \frac{\partial v^{(2)}}{\partial \xi} \right] , \tag{16}$$

$$\sigma_{\eta\eta}^{(2)} = \frac{\mu_2}{P_o}\left[(N^2 - 2)\frac{\partial u^{(2)}}{\partial\xi} + v^2\frac{\partial V^{(2)}}{\partial\eta}\right] . \tag{17}$$

The boundary conditions, the regularity conditions, and the continuity conditions are correspondingly

$$\sigma_{\xi\eta}^{(1)} = \mu_f P'_{(\xi)} \qquad (|\xi| \leq 1 , \eta = 0) , \tag{18}$$

$$\sigma_{\eta\eta}^{(1)} = \begin{cases} 0 & (|\xi| > 1 , \eta = 0) \\ -P'_{(\xi)} & (|\xi| \leq 1 , \eta = 0) \end{cases} , \tag{19}$$

$$\sigma_{\xi\xi}, \sigma_{\xi\eta}, \sigma_{\eta\eta}, u, V \to 0 \qquad (0 \leq \eta \leq D , |\xi| \to \infty) , \tag{20}$$

$$\sigma_{\xi\xi}, \sigma_{\xi\eta}, \sigma_{\eta\eta}, u, V \to 0 \qquad (\xi^2 + \eta^2 \to \infty) , \tag{21}$$

$$\sigma_{\xi\eta}^{(1)} = \sigma_{\xi\eta}^{(2)} , \ \sigma_{\eta\eta}^{(1)} = \sigma_{\eta\eta}^{(2)} , \ u^{(1)} = u^{(2)} , \ V^{(1)} = V^{(2)} \ (\eta = D) . \tag{22}$$

2.1.2 Temperature field

$$\frac{\partial^2\phi^{(1)}}{\partial\xi^2} + \frac{\partial^2\phi^{(1)}}{\partial\eta^2} = R_1 \frac{\partial\phi^{(1)}}{\partial\xi} , \tag{23}$$

$$\frac{\partial^2\phi^{(2)}}{\partial\xi^2} + \frac{\partial^2\phi^{(2)}}{\partial\eta^2} = R_2 \frac{\partial\phi^{(2)}}{\partial\xi} . \tag{24}$$

The boundary conditions, the continuity conditions, and the regularity conditions are

$$-\frac{\partial\phi^{(1)}}{\partial\eta} = Q^{*}(\xi) = \begin{cases} Q(\xi) & (|\xi| \le 1 \ , \ \eta = 0) \\ 0 & (|\xi| > 1 \ , \ \eta = 0) \end{cases} , \tag{25}$$

$$\frac{\partial\phi^{(1)}}{\partial\eta} = \beta \frac{\partial\phi^{(2)}}{\partial\eta} \qquad (\eta = D) \ , \tag{26}$$

$$\phi^{(1)} = \phi^{(2)} \qquad (\eta = D) \ , \tag{27}$$

$$\phi^{(1)},\phi^{(2)} \to 0 \qquad (\xi^2 + \eta^2 \to \infty) \ . \tag{28}$$

2.1.3. <u>Thermal stress field</u>

In the surface layer

$$(I^2 - M^2)\frac{\partial^2 u^{(1)}}{\partial\xi^2} + (I^2 - J^2)\frac{\partial^2 v^{(1)}}{\partial\xi\partial\eta} + J^2\frac{\partial^2 u^{(1)}}{\partial\eta^2} - \frac{b^{*2}\gamma_1}{c_2^2}\frac{\partial\phi^{(1)}}{\partial\xi} = 0 \ , \tag{29}$$

$$(J^2 - M^2)\frac{\partial^2 v^{(1)}}{\partial\xi^2} + (I^2 - J^2)\frac{\partial^2 u^{(1)}}{\partial\xi\partial\eta} + I^2\frac{\partial^2 v^{(1)}}{\partial\eta^2} - \frac{b^{*2}\gamma_1}{c_2^2}\frac{\partial\phi^{(1)}}{\partial\eta} = 0 \ , \tag{30}$$

$$\sigma_{\xi\xi}^{(1)} = \delta\frac{\mu_2}{P_o}\left[I^2\frac{\partial u^{(1)}}{\partial\xi} + (I^2 - 2J^2)\frac{\partial v^{(1)}}{\partial\eta} - \frac{b^{*2}\gamma_1}{c_2^2}\phi^{(1)}\right] , \tag{31}$$

$$\sigma_{\xi\eta}^{(1)} = \delta J^2\frac{\mu_2}{P_o}\left[\frac{\partial u^{(1)}}{\partial\eta} + \frac{\partial v^{(1)}}{\partial\xi}\right] , \tag{32}$$

$$\sigma_{\eta\eta}^{(1)} = \delta \frac{\mu_2}{P_o}\left[(I^2 - 2J^2) \frac{\partial u^{(1)}}{\partial \xi} + I^2 \frac{\partial V^{(1)}}{\partial \eta} - \frac{b^{*2}\gamma_1}{c_2^2} \phi^{(1)} \right] . \tag{33}$$

In the substrate region

$$(N^2 - M^2) \frac{\partial^2 u^{(2)}}{\partial \xi^2} + (N^2 - 1) \frac{\partial^2 V^{(2)}}{\partial \xi \partial \eta} + \frac{\partial^2 u^{(2)}}{\partial \eta^2} - \frac{b^2 \gamma_2}{c_2^2} \frac{\partial \phi^{(2)}}{\partial \xi} = 0 , \tag{34}$$

$$(1 - M^2) \frac{\partial^2 V^{(2)}}{\partial \xi^2} + (N^2 - 1) \frac{\partial^2 u^{(2)}}{\partial \xi \partial \eta} + N^2 \frac{\partial^2 V^{(2)}}{\partial \eta^2} - \frac{b^2 \gamma_2}{c_2^2} \frac{\partial \phi^{(2)}}{\partial \eta} = 0 , \tag{35}$$

$$\sigma_{\xi\xi}^{(2)} = \frac{\mu_2}{P_o}\left[N^2 \frac{\partial u^{(2)}}{\partial \xi} + (N^2 - 2) \frac{\partial V^{(2)}}{\partial \eta} - \frac{b^2\gamma_2}{c_2^2} \phi^{(2)} \right] , \tag{36}$$

$$\sigma_{\xi\eta}^{(2)} = \frac{\mu_2}{P_o}\left[\frac{\partial u^{(2)}}{\partial \eta} + \frac{\partial V^{(2)}}{\partial \xi} \right] , \tag{37}$$

$$\sigma_{\eta\eta}^{(2)} = \frac{\mu_2}{P_o}\left[(N^2 - 2) \frac{\partial u^{(2)}}{\partial \xi} + N^2 \frac{\partial V^{(2)}}{\partial \eta} - \frac{b^2\gamma_2}{c_2^2} \phi^{(2)} \right] . \tag{38}$$

The boundary conditions, the continuity conditions, and the regularity conditions are

$$\sigma_{\xi\eta}^{(1)} = 0 \qquad (\eta = 0) , \tag{39}$$

$$\sigma_{\eta\eta}^{(1)} = 0 \qquad (\eta = 0) , \tag{40}$$

$$\sigma_{\xi\xi}, \sigma_{\xi\eta}, \sigma_{\eta\eta}, u, V \rightarrow 0 \qquad (0 \leq \eta \leq D , \quad \xi \rightarrow \infty) . \tag{41}$$

$$\sigma_{\xi\xi}, \sigma_{\xi\eta}, \sigma_{\eta\eta}, u, v \to 0 \qquad (\xi^2 + \eta^2 \to \infty) \ , \tag{42}$$

$$\sigma_{\xi\eta}^{(1)} = \sigma_{\xi\eta}^{(2)} \ , \ \sigma_{\eta\eta}^{(1)} = \sigma_{\eta\eta}^{(2)} \ , \ u^{(1)} = u^{(2)} \ , \ v^{(1)} = v^{(2)} \qquad (\eta = D) \ . \tag{43}$$

2.2 General solution

The Fourier transform [7]

$$F\{\ \} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \{\ \} \, e^{is\xi} \, d\xi$$

is used, denoting the transformed quantities by a superposed tilda ~ , then letting $\bar{u} = i\tilde{u}$, $\bar{v} = \tilde{v}$, $\zeta = s\eta$, $\bar{\sigma}_{\xi\xi} = \tilde{\sigma}_{\xi\xi}/s$, $\bar{\sigma}_{\xi\eta} = i\tilde{\sigma}_{\xi\eta}/s$, $\bar{\sigma}_{\eta\eta} = \tilde{\sigma}_{\eta\eta}/s$, $\bar{P} = \tilde{P}/s$, $\bar{\Phi} = \tilde{\Phi}/s$, $\bar{Q}^* = \tilde{Q}^*/s$, and $' = \frac{d}{d\zeta} = \frac{1}{s}\frac{d}{d\eta}$. We can obtain the analytical solutions in the transformed space.

2.2.1 The analytical solutions of mechanical stress field

Equations (8) through (17) become

$$J^2 \, \bar{u}^{(1)''} - (I^2 - M^2) \, \bar{u}^{(1)} + (I^2 - J^2) \, \bar{v}^{(1)'} = 0 \ , \tag{44}$$

$$I^2 \, \bar{v}^{(1)''} - (J^2 - M^2) \, \bar{v}^{(1)} - (I^2 - J^2) \, \bar{u}^{(1)'} = 0 \ , \tag{45}$$

$$\bar{\sigma}_{\xi\xi}^{(1)} = \delta \frac{\mu_2}{P_o} [-I^2 \, \bar{u}^{(1)} + (I^2 - 2J^2) \, \bar{v}^{(1)}] \ , \tag{46}$$

$$\bar{\sigma}_{\xi\eta}^{(1)} = \delta J^2 \frac{\mu_2}{P_o} (\bar{u}^{(1)'} + \bar{v}^{(1)}) \ , \tag{47}$$

$$\bar{\sigma}_{\eta\eta}^{(1)} = \delta \frac{\mu_2}{P_o} [I^2 \bar{V}^{(1)'} - (I^2 - 2J^2) \bar{u}^{(1)}] , \tag{48}$$

$$\bar{u}^{(2)''} - (N^2 - M^2) \bar{u}^{(2)} + (N^2 - 1) \bar{V}^{(2)'} = 0 , \tag{49}$$

$$N^2 \bar{V}^{(2)''} - (1 - M^2) \bar{V}^{(2)} - (N^2 - 1) \bar{u}^{(2)'} = 0 , \tag{50}$$

$$\bar{\sigma}_{\xi\xi}^{(2)} = \frac{\mu_2}{P_o} [-N^2 \bar{u}^{(2)} + (N^2 - 2) \bar{V}^{(2)'}] , \tag{51}$$

$$\bar{\sigma}_{\xi\eta}^{(2)} = \frac{\mu_2}{P_o} (\bar{u}^{(2)'} + \bar{V}^{(2)}) , \tag{52}$$

$$\bar{\sigma}_{\eta\eta}^{(2)} = \frac{\mu_2}{P_o} [N^2 \bar{V}^{(2)'} - (N^2 - 2) \bar{u}^{(2)}] . \tag{53}$$

The boundary conditions, the regularity conditions and the continuity conditions are also in transformed expressions

$$\bar{\sigma}_{\xi\eta}^{(1)} = i\mu_f \bar{P} = \frac{i\mu_f}{\sqrt{2\pi}\, s} \int_{-\infty}^{\infty} P^*(\xi)\, e^{is\xi}\, d\xi \text{ at } \zeta = 0 \tag{54}$$

$$\bar{\sigma}_{\eta\eta}^{(1)} = - \bar{P} = - \frac{1}{\sqrt{2\pi}\, s} \int_{-\infty}^{\infty} P^*(\xi)\, e^{is\xi}\, d\xi \text{ at } \zeta = 0 \tag{55}$$

where $P^*(\xi) = \begin{cases} P'(\xi) & (|\xi| \leq 1) \\ 0 & (|\xi| \geq 1) \end{cases}$

$$\bar{\sigma}_{\xi\xi},\ \bar{\sigma}_{\xi\eta},\ \bar{\sigma}_{\eta\eta},\ \bar{u},\ \bar{V} \to 0 \qquad (\zeta \to \infty) \tag{56}$$

$$\bar{\sigma}_{\xi\eta}{}^{(1)} = \bar{\sigma}_{\xi\eta}{}^{(2)},\ \ \bar{\sigma}_{\eta\eta}{}^{(1)} = \bar{\sigma}_{\eta\eta}{}^{(2)},\ \bar{u}^{(1)} = \bar{u}^{(2)}\ ,\ \bar{V}^{(1)} = \bar{V}^{(2)} \quad (\zeta = SD) \tag{57}$$

From equations (49) and (50) we can get

$$\bar{u}^{(2)} = f^{-2}\ (f^2 - g^2)^{-1}\ [(1 - f^2)\ \bar{V}^{(2)'''} + (f^4 - g^2)\bar{V}^{(2)'}] \tag{58}$$

$$\bar{V}^{(2)^{(4)}} - (f^2 + g^2)\ \bar{V}^{(2)''} + f^2 g^2\ \bar{V}^{(2)} = 0 \tag{59}$$

where

$$f = (1 - M^2/N^2)^{1/2}\ ,\quad g = (1 - M^2)^{1/2}\ .$$

For the subsonic case

$$M = v/c^2 < 1,\quad N = c_1/c_2 > 1$$

$$f = (1 - M^2/N^2)^{1/2} = (1 - v^2/c_1{}^2)^{1/2} > 0$$

$$g = (1 - M)^{1/2} = (1 - v^2/c_2{}^2)^{1/2} > 0$$

and $c_1 > c_2$ so that $f > g$.

Equations (44) and (45) yield

$$\bar{u}^{(1)} = (I^2 - M^2)^{-1}\ (I^2 - J^2)^{-1}\ \ I^2 J^2\ \bar{V}^{(1)'''}$$
$$+ [(I^2 - J^2)^2 - J^2(J^2 - M^2)]\ \bar{V}^{(1)'} \tag{60}$$

and

$$\bar{V}^{(1)^{(4)}} - [(1 - M^2/J^2) + (1 - M^2/I^2)]\bar{V}^{(1)''}$$
$$+ (1 - M^2/I^2)(1 - M^2/J^2)\bar{V}^{(1)} = 0 \tag{61}$$

Equation (61) has a characteristic equation

$$\lambda^4 - [(1 - M^2/J^2) + (1 - M^2/I^2)]\lambda^2 + (1 - M^2/I^2)(1 - M^2/J^2) = 0 \tag{62}$$

in which λ is the characteristic value of $\bar{V}^{(1)} = e^{\lambda\zeta}$.

For the subsonic case

$$M^2/J^2 < 1,\ M^2/I^2 < 1,\ \text{and } I > J.$$

with $j = (1 - M^2/I^2)^{1/2}$, $k = (1 - M^2/J^2)^{1/2}$, then applying the boundary conditions, we can obtain the solutions for $s > 0$.

$$\bar{V}^{(1)} = a_1^* \cosh j\eta s + b_1^* \sinh j\eta s + c_1^* \cosh k\eta s + d_1^* \sinh k\eta s \ , \tag{63}$$

$$\bar{u}^{(1)} = j^{-1} (a_1^* \sinh j\eta s + b_1^* \cosh j\eta s) + k(c_1^* \sinh k\eta s + d_1^* \cosh k\eta s), \tag{64}$$

$$\bar{\sigma}_{\xi\xi}^{(1)} = \frac{\mu_2}{P_o} [-\delta(M^2 j^{-1} + 2J^2 j)\ (a_1^* \sinh j\eta s + b_1^* \cosh j\eta s) - 2\delta J^2 k(c_1^* \sinh k\eta s + d_1^* \cosh k\eta s)] \ , \tag{65}$$

$$\bar{\sigma}_{\xi\eta}^{(1)} = \frac{\mu_2}{P_o} [2\delta J^2(a_1^* \cosh j\eta s + b_1^* \sinh j\eta s) + \delta J^2(1 + k^2) (c_1^* \cosh k\eta s + d_1^* \sinh k\eta s)] \ , \tag{66}$$

$$\bar{\sigma}_{\eta\eta}^{(1)} = \frac{\mu_2}{P_o} [\delta J^2 j^{-1}(1 + k^2)\ (a_1^* \sinh j\eta s + b_1^* \cosh j\eta s) + 2\delta J^2 k(c_1^* \sinh k\eta s + d_1^* \cosh k\eta s)] \ , \tag{67}$$

$$\bar{v}^{(2)} = A_1^* e^{-f\eta s} + B_1^* e^{-g\eta s} , \qquad (68)$$

$$\bar{u}^{(2)} = -f^{-1} A_1^* e^{-f\eta s} - gB_1^* e^{-g\eta s} , \qquad (69)$$

$$\bar{\sigma}_{\xi\xi}^{(2)} = \frac{\mu_2}{P_o} [f^{-1}(M^2 + 2f^2) A_1^* e^{-f\eta s} + 2gB_1^* e^{-g\eta s}] , \qquad (70)$$

$$\bar{\sigma}_{\xi\eta}^{(2)} = \frac{\mu_2}{P_o} [2A_1^* e^{-f\eta s} + (1 + g^2) B_1^* e^{-g\eta s}] , \qquad (71)$$

$$\bar{\sigma}_{\eta\eta}^{(2)} = \frac{\mu_2}{P_o} [-f^{-1}(1 + g^2) A_1^* e^{-f\eta s} - 2gB_1^* e^{-g\eta s}] . \qquad (72)$$

For $s < 0$ we can get a similar set of solutions. Where a_1^*, b_1^*, c_1^*, d_1^* A_1^*, and B_1^* are functions of s and depend on the pressure distribution profile $\bar{P}$, that

$$\bar{P} = \frac{2 \sin s}{\sqrt{2\pi}\ s^2}$$

for uniform pressure and

$$\bar{P} = \frac{6}{\sqrt{2\pi}\ s^3} \left(\frac{\sin s}{s} - \cos s\right)$$

for parabolic pressure.

Now enforcing the boundary conditions (54), (55), and (57), the six constants A_1^*, B_1^*, a_1^*, b_1^*, c_1^*, and d_1^* are interrelated by six algebraic equations

$$2a_1^* + (1 + k^2)\, c_1^* = \frac{i\mu_f}{\delta J^2}\, \bar{P} \tag{73}$$

$$j^{-1}\, (1 + k^2)\, b_1^* + 2kd_1^* = -\frac{\bar{P}}{\delta J^2} \tag{74}$$

$$j^{-1}s_j\, a_1^* + j^{-1}c_j\, b_1^* + ks_k c_1^* + kc_k d_1^* + f^{-1}e^{-fsD}\, A_1^* + ge^{-gsD} B_1^* = 0 \tag{75}$$

$$c_j a_1^* + s_j b_1^* + c_k c_1^* + s_k d_1^* - e^{-fsD}\, A_1^* - e^{-gsD}\, B_1^* = 0 \tag{76}$$

$$2c_j a_1^* + 2s_j b_1^* + (1 + k^2)c_k c_1^* + (1 + k^2)s_k d_1^* - \frac{2e^{-fsD}}{\delta J^2}\, A_1^*$$

$$- \frac{(1 + g^2)e^{-gsD}}{\delta J^2}\, B_1^* = 0 \tag{77}$$

$$j^{-1}(1 + k^2)s_j a_1^* + j^{-1}(1 + k^2)c_j b_1^* + 2ks_k c_1^* + 2kc_k\, d_1^*$$

$$+ \frac{(1 + g^2)\, e^{-fsD}}{f\delta J^2}\, A_1^* + \frac{2ge^{-gsD}}{\delta J^2}\, B_1^* = 0 \tag{78}$$

In the above

s_j = sinh (jsD)

c_j = cosh (jsD)

s_k = sinh (ksD)

c_k = cosh (ksD)

represent the six algebraic equations in matrix form, then solve it by Cramer's rule.

$$
\begin{bmatrix}
2 & 0 & (1+k^2) & 0 & 0 & 0 \\
0 & j^{-1}(1+k^2) & 0 & 2k & 0 & 0 \\
j^{-1}s_j & j^{-1}c_j & ks_k & kc_k & f^{-1}e^{-fsD} & ge^{-gsD} \\
c_j & s_j & c_k & s_k & -e^{-fsD} & -e^{-gsD} \\
2c_j & 2s_j & (1+k^2)c_k & (1+k^2)s_k & -\dfrac{2e^{-fsD}}{\delta J^2} & -\dfrac{(1+g^2)e^{-gsD}}{\delta J^2} \\
j^{-1}(1+k^2)s_j & j^{-1}(1+k^2)c_j & 2ks_k & 2kc_k & \dfrac{(1+g^2)e^{-fsD}}{f\delta J^2} & \dfrac{2ge^{-gsD}}{\delta J^2}
\end{bmatrix}
\begin{bmatrix} a_1^* \\ b_1^* \\ c_1^* \\ d_1^* \\ A_1^* \\ B_1^* \end{bmatrix}
$$

$$
= \begin{bmatrix} \dfrac{i\mu_f\bar{P}}{\delta J^2} \\ -\dfrac{\bar{P}}{\delta J^2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
a_1^* = \frac{\Delta_1}{\Delta},\quad b_1^* = \frac{\Delta_2}{\Delta},\quad c_1^* = \frac{\Delta_3}{\Delta},\quad d_1^* = \frac{\Delta_4}{\Delta},\quad A_1^* = \frac{\Delta_5}{\Delta},\quad B_1^* = \frac{\Delta_6}{\Delta}
$$

The expressions of Δ, Δ_1, Δ_2, Δ_3, Δ_4, Δ_5, and Δ_6 are given in Appendix I.

2.2.2 The analytical solutions of temperature field

Equations (23) and (24) in the transformed space became

$$\bar{\phi}^{(1)''} - (1 - iR_1/s)\, \bar{\phi}^{(1)} = 0 \; , \tag{79}$$

$$\bar{\phi}^{(2)''} - (1 - iR_2/s)\, \bar{\phi}^{(2)} = 0 \; . \tag{80}$$

Applying boundary conditions we can get the solutions for $s > 0$.

$$\bar{\phi}^{(1)} = A_1^{'}\, e^{-\sqrt{1 - iR_1/s}\; \eta s} + B_1^{'}\, e^{\sqrt{1 - iR_1/s}\; \eta s} \; , \tag{81}$$

$$\bar{\phi}^{(2)} = A_2^{'}\, e^{-\sqrt{1 - iR_2/s}\; \eta s} \tag{82}$$

where $A_1^{'}$, $B_1^{'}$, and $A_2^{'}$ are functions of s and depend on the heat input profile $\bar{Q}^*$, that

$$\bar{Q}^* = \frac{2 \sin s}{\sqrt{2\pi}\, s^2}$$

for the uniform pressure and

$$\bar{Q}^* = \frac{6}{\sqrt{2\pi}\, s^3}\left(\frac{\sin s}{s} - \cos s\right)$$

for parabolic pressure.

For $s < 0$, Equations (81) and (82) may be used, provided s is replaced by $n = -s$ and a negative sign is added to $\bar{\phi}^{(2)}$.

Apply the boundary conditions and the continuity condition at the surface layer/substrate interface, $A_1^{'}$, $B_1^{'}$, and $A_2^{'}$ can be readily solved.

$$A_1' = \frac{\bar{Q}^*}{s} \frac{F + \beta G}{F^2(1 - e^{-2FsD}) + \beta FG(1 + e^{-2FsD})}$$

$$B_1' = \frac{\bar{Q}^*}{s} \frac{(F - \beta G)\, e^{-2FsD}}{F^2(1 - e^{-2FsD}) + \beta FG(1 + e^{-2FsD})}$$

$$A_2' = \frac{\bar{Q}^*}{s} \frac{2e^{(-F+G)sD}}{F(1 - e^{-2FsD}) + \beta G(1 + e^{-2FsD})}$$

where $F = (1 - iR_1/s)^{1/2}$, $G = (1 - iR_2/s)^{1/2}$

2.2.3 The analytical solutions of thermal stress field

Equations (29) through (38) become

$$J^2\bar{u}^{(1)''} - (I^2 - M^2)\,\bar{u}^{(1)} + (I^2 - J^2)\,\bar{v}^{(1)'} = \frac{b^{*2}\gamma_1}{c_2^2}\,\bar{\phi}^{(1)} , \tag{83}$$

$$I^2\bar{v}^{(1)''} - (J^2 - M^2)\,\bar{v}^{(1)} - (I^2 - J^2)\,\bar{u}^{(1)'} = \frac{b^{*2}\gamma_1}{c_2^2}\,\bar{\phi}^{(1)'} , \tag{84}$$

$$\bar{\sigma}_{\xi\xi}^{(1)} = \delta\,\frac{\mu_2}{P_0}\left[-I^2\,\bar{u}^{(1)} + (I^2 - 2J^2)\,\bar{v}^{(1)'} - \frac{b^{*2}\gamma_1}{c_2^2}\,\bar{\phi}^{(1)}\right] , \tag{85}$$

$$\bar{\sigma}_{\xi\eta}^{(1)} = \delta J^2\,\frac{\mu_2}{P_0}\,(\bar{u}^{(1)'} + \bar{v}^{(1)}) . \tag{86}$$

$$\bar{\sigma}_{\eta\eta}^{(1)} = \delta\,\frac{\mu_2}{P_0}\left[I^2\,\bar{v}^{(1)'} - (I^2 - 2J^2)\,\bar{u}^{(1)} - \frac{b^{*2}\gamma_1}{c_2^2}\,\bar{\phi}^{(1)}\right] , \tag{87}$$

$$\bar{u}^{(2)''} - (N^2 - M^2)\,\bar{u}^{(2)} + (N^2 - 1)\,\bar{v}^{(2)'} = \frac{b^2\gamma_2}{c_2^2}\,\bar{\phi}^{(2)}\ , \tag{88}$$

$$N^2\,\bar{v}^{(2)''} - (1 - M^2)\,\bar{v}^{(2)} - (N^2 - 1)\,\bar{u}^{(2)'} = \frac{b^2\gamma_2}{c_2^2}\,\bar{\phi}^{(2)'}\ , \tag{89}$$

$$\bar{\sigma}_{\xi\xi}^{(2)} = \frac{\mu_2}{P_o}\left[-N^2\,\bar{u}^{(2)} + (N^2 - 2)\,\bar{v}^{(2)'} - \frac{b^2\gamma_2}{c_2^2}\,\bar{\phi}^{(2)}\right]\ , \tag{90}$$

$$\bar{\sigma}_{\xi\eta}^{(2)} = \frac{\mu_2}{P_o}\,(\bar{u}^{(2)'} + \bar{v}^{(2)}) \tag{91}$$

$$\bar{\sigma}_{\eta\eta}^{(2)} = \frac{\mu_2}{P_o}\left[-\,(N^2 - 2)\,\bar{u}^{(2)} + N^2\,\bar{v}^{(2)'} - \frac{b^2\gamma_2}{c_2^2}\,\bar{\phi}^{(2)}\right] \tag{92}$$

The boundary conditions, the regularity conditions and the continuity conditions are

$$\bar{\sigma}_{\xi\eta}^{(1)} = 0 \qquad \text{at } \zeta = 0 \tag{93}$$

$$\bar{\sigma}_{\eta\eta}^{(1)} = 0 \qquad \text{at } \zeta = 0 \tag{94}$$

$$\bar{\sigma}_{\xi\xi},\ \bar{\sigma}_{\xi\eta},\ \bar{\sigma}_{\eta\eta},\ \bar{u},\ \bar{v} \to 0 \qquad \text{when } \zeta \to \infty \tag{95}$$

$$\bar{\sigma}_{\xi\eta}^{(1)} = \bar{\sigma}_{\xi\eta}^{(2)},\quad \bar{\sigma}_{\eta\eta}^{(1)} = \bar{\sigma}_{\eta\eta}^{(2)},\quad \bar{u}^{(1)} = \bar{u}^{(2)}\ ,\ \text{and } \bar{v}^{(1)} = \bar{v}^{(2)} \ \text{at } \zeta = sD \tag{96}$$

where $\bar{\phi}^{(1)}$, $\bar{\phi}^{(1)'}$, $\bar{\phi}^{(2)}$ and $\bar{\phi}^{(2)'}$ come from temperature field,

$$\bar{\phi}^{(1)'} = F(-A_1^{'}\,e^{-F\zeta} + B_1^{'}\,e^{F\zeta})$$

$$\bar{\phi}^{(2)'} = - G_2 A_2' \ e^{-G\zeta} .$$

A_1', B_1', and A_2' are the same as in the temperature field. Equations (88) and (89) yield

$$\bar{u}^{(2)} = f^{-2} \ (f^2 - g^2)^{-1} \ [(1 - f^2) \ \bar{V}^{(2)'''} + (f^4 - g^2) \ \bar{V}^{(2)'}] \quad (97)$$

$$- E_1 \phi^{(2)} - E_2 \phi^{(2)''}$$

$$\bar{V}^{(2)^{(4)}} - (f^2 + g^2) \ \bar{V}^{(2)''} + f^2 g^2 \ \bar{V}^{(2)} = E_3 \bar{\phi}^{(2)'} - E_4 \ \bar{\phi}^{(2)'''} \quad (98)$$

where $E_1 = \dfrac{1}{(N^2 - M^2)} \dfrac{b^2 \gamma_2}{c_2{}^2}$, $E_2 = \dfrac{1}{(N^2 - M^2)(N^2 - 1)} \dfrac{b^2 \gamma_2}{c_2{}^2}$

$$E_3 = \frac{-g^2}{N^2} \frac{b^2 \gamma_2}{c_2{}^2} \ , \quad E_4 = \frac{1}{N^2} \frac{b^2 \gamma_2}{c_2{}^2}$$

$$\bar{\phi}^{(2)''} = G^2 A_2' \ e^{-G\zeta}$$

and $\bar{\phi}^{(2)'''} = -G^3 A_2' \ e^{-G\zeta}$

Equation (98) is a fourth order nonhomogeneous ordinary differential equation, where the nonhomogeneous part comes from temperature field.

The complementary solution of equation (98) is

$$\bar{V}^{(2)^c} = A_2^* e^{-f\zeta} + B_2^* e^{f\zeta} + c_2^* \ e^{-g\zeta} + D_2^* e^{g\zeta} \quad (99)$$

Assume the particular solution of equation (98) is

$$\bar{V}^{(2)^p} = A_3^* e^{-G\zeta} \quad (100)$$

Substituting (100) into (98), we can obtain A_3^* .

$$A_3^* = \frac{\bar{Q}^*}{s} \frac{(-E_3G + E_4G^3)}{[G^4 - (f^2 + g^2)G^2 + f^2g^2]} \frac{2e^{(-F+G)sD}}{[F(1 - e^{-2FsD}) + \beta G(1 + e^{-2FsD})]}$$

The solution of equation (98) is the combination of the complementary solution and the particular solution

$$\bar{v}^{(2)} = A_2^* e^{-f\zeta} + B_2^* e^{-g\zeta} + c_2^* e^{f\zeta} + D_2^* e^{g\zeta} + A_3^* e^{-G\zeta} \tag{101}$$

Equation (101) together with the regularity condition (95) yields a solution for $s > 0$.

$$\bar{v}^{(2)} = A_2^* e^{-f\zeta} + B_2^* e^{-g\zeta} + A_3^* e^{-G\zeta} \tag{102}$$

Substituting (102) into (97) $\bar{u}^{(2)}$ can be obtained.

$$\bar{u}^{(2)} = -f^{-1} A_2^* e^{-f\zeta} - gB_2^* e^{-g\zeta} + E_5 e^{-G\zeta} \tag{103}$$

where $E_5 = -\Big\{ f^{-2}(f^2 - g^2)^{-1}[(1 - f^2)G^3 + (f^4 - g^2)G]A_3{}^*$

$+ (E_1 + E_2G^2)A_2{}' \Big\}$

Equations (102) and (103) together with (90), (91) and (92) yield

$$\overline{\sigma_{\zeta\zeta}}^{(2)} = \frac{\mu_2}{P_o} [f^{-1}(M^2 + 2f^2)A_2^* e^{-f\zeta} + 2gB_2^* e^{-g\zeta} + E_6 e^{-G\zeta}] \tag{104}$$

$$\overline{\sigma_{\xi\eta}}^{(2)} = \frac{\mu_2}{P_o} [2A_2^* e^{-f\zeta} + (1 + g^2)\ B_2{}^*\ e^{-g\zeta} + E_7\ e^{-G\zeta}] \tag{105}$$

$$\overline{\sigma_{\eta\eta}}^{(2)} = \frac{\mu_2}{P_o} [-f^{-1}(1 + g^2)A_2^* e^{-f\zeta} - 2gB_2^* e^{-g\zeta} + E_8 e^{-G\zeta}] \tag{106}$$

where $E_6 = - N^2E_5 - (N^2 - 2)\ GA_3^* - \frac{b^2\gamma_2}{c_2^{\ 2}} A_2'$

$$E_7 = - E_5G + A_3^*$$

$$E_8 = - (N^2 - 2)\ E_5 - N^2GA_3^* - \frac{b^2\gamma_2}{c_2^{\ 2}} A_2'$$

Equations (83) and (84) give

$$\overline{u^{(1)}} = \frac{-1}{j^2p}\ \overline{v^{(1)}}''' + (j^{-2} + p^{-1})\ \overline{v^{(1)}}' + \frac{1}{I^2j^2p}\ \frac{b^{*2}\gamma_1}{c_2^{\ 2}}\ \overline{\phi^{(1)}}''$$

$$- \frac{1}{I^2j^2}\ \frac{b^{*2}\gamma_1}{c_2^{\ 2}}\ \overline{\phi^{(1)}} \qquad (107)$$

$$\overline{v^{(1)}}^{(4)} - (j^2 + k^2)\ \overline{v^{(1)}}'' + j^2k^2\ \overline{v^{(1)}} - E_9\ \overline{\phi^{(1)}}''' + E_{10}\ \overline{\phi^{(1)}}' = 0$$

where $P = 1 - I^2/J^2$ (108)

$$E_9 = \frac{1}{I^2}\ \frac{b^{*2}\gamma_1}{c_2^{\ 2}}$$

$$E_{10} = \frac{k^2}{I^2}\ \frac{b^{*2}\gamma_1}{c_2^{\ 2}}$$

The complementary solution of equation (108) is

$$\overline{v^{(1)}}^{c} = a_2^* \cosh j\zeta + b_2^* \sinh j\zeta + c_2^* \cosh k\zeta + d_2^* \sinh k\zeta \tag{109}$$

Assume the particular solution of equation (108) is

$$\overline{v^{(1)}}^{p} = A_4^* e^{-F\zeta} + A_5^* e^{F\zeta} \tag{110}$$

Substituting (110) into (108) we can obtain A_4^* and A_5^*.

$$A_4^* = \frac{(-F^3 E_9 + E_{10} F)\, A_1'}{F^4 - (j^2 + k^2)F^2 + j^2 k^2}$$

$$A_5^* = \frac{(F^3 E_9 - F\, E_{10}) B_1'}{F^4 - (j^2 + k^2)F^2 + j^2 k^2}$$

The solution of equation (108) is

$$\overline{v^{(1)}} = a_2^* \cosh j\zeta + b_2^* \sinh j\zeta + c_2^* \cosh k\zeta + d_2^* \sinh k\zeta$$

$$+ A_4^* e^{-F\zeta} + A_5^* e^{F\zeta} \tag{111}$$

Substituting $\overline{v^{(1)}}$ into (107) gives

$$\overline{u^{(1)}} = j^{-1} (a_2^* \sinh j\zeta + b_2^* \cosh j\zeta) + k(c_2^* \sinh k\zeta + d_2^* \cosh k\zeta)$$

$$+ E_{11}\, e^{-F\zeta} + E_{12}\, e^{F\zeta} \tag{112}$$

where $E_{11} = \dfrac{F^3}{j^2 P} A_4^* - (j^{-2} + P^{-1}) F A_4^* + \dfrac{1}{I^2 j^2 P} \dfrac{b^{*2} \gamma_1}{c_2^{\,2}} F^2 A_1'$

$$- \frac{1}{I^2 j^2} \frac{b^{*2} \gamma_1}{c_2^{\,2}} A_1'$$

$$E_{12} = - \frac{F^3}{j^2 p} A_5^* + (j^{-2} + p^{-1}) FA_5^* + \frac{1}{I^2 j^2 p} \frac{b^{*2} \gamma_1}{c_2^{\,2}} F^2 B_1'$$

$$- \frac{1}{I^2 j^2} \frac{b^{*2} \gamma_1}{c_2^{\,2}} B_1'$$

Equations (111) and (112) together with (85), (86), and (87) yield

$$\overline{\sigma_{\xi\xi}}^{(1)} = \delta \frac{\mu_2}{P_o} [- (M^2 j^{-1} + 2J^2 j)(a_2^* \sinh j\zeta + b_2^* \cosh j\zeta)$$

$$- 2J^2 k(c_2^* \sinh k\zeta + d_2^* \cosh k\zeta) + E_{13} e^{-F\zeta} + E_{14} e^{F\zeta}] \qquad (113)$$

$$\overline{\sigma_{\xi\eta}}^{(1)} = \delta J^2 \frac{\mu_2}{P_o} [2(a_2^* \cosh j\zeta + b_2^* \sinh j\zeta) + (1 + k^2)(c_2^* \cosh k\zeta$$

$$+ d_2^* \sinh k\zeta) + E_{15}\, e^{-F\zeta} + E_{16}\, e^{F\zeta}] \qquad (114)$$

$$\overline{\sigma_{\eta\eta}}^{(1)} = \delta \frac{\mu_2}{P_o} \Big\{ [J^2 j^{-1} (1 + k^2)](a_2^* \sinh j\zeta + b_2^* \cosh j\zeta)$$

$$+ 2J^2 k(c_2^* \sinh k\zeta + d_2^* \cosh k\zeta) + E_{17} e^{-F\zeta} + E_{18} e^{F\zeta}] \Big\} \qquad (115)$$

where

$$E_{13} = - I^2 E_{11} - (I^2 - 2J^2) FA_4^* - \frac{b^{*2} \gamma_1}{c_2^{\,2}} A_1'$$

$$E_{14} = - I^2 E_{12} + (I^2 - 2J^2) FA_5^* - \frac{b^{*2} \gamma_1}{c_2^{\,2}} B_1'$$

$$E_{15} = - FE_{11} + A_4^*$$

$$E_{16} = FE_{12} + A_5^*$$

$$E_{17} = - I^2FA_4^* - (I^2 - 2J^2)E_{11} - \frac{b^{*2}\gamma_1}{c_2^{\;2}} A_1',$$

$$E_{18} = I^2FA_5^* - (I^2 - 2J^2)E_{12} - \frac{b^{*2}\gamma_1}{c_2^{\;2}} B_1'$$

Applying the boundary conditions and the continuity conditions at the surface layer/substrate interface, the six constants A_2^*, B_2^*, a_2^*, b_2^*, c_2^* , and d_2^* can be solved by six algebraic equations

$$2a_2^* + (1 + k^2)c_2^* = E_{19}$$

$$j^{-1}(1 + k^2)b_2^* + 2kd_2^* = E_{20}$$

$$j^{-1}s_ja_2^* + j^{-1}c_jb_2^* + ks_kc_2^* + kc_kd_2^* + f^{-1}A_2^*e^{-fsD} + gB_2^*e^{-gsD} = E_{21}$$

$$c_ja_2^* + s_jb_2^* + c_kc_2^* + s_kd_2^* - A_2^*e^{-fsD} - B_2^*e^{-gsD} = E_{22}$$

$$2c_ja_2^* + 2s_jb_2^* + (1 + k^2)c_kc_2^* + (1 + k^2)s_kd_2^* - \frac{2e^{-fsD}}{\delta J^2} A_2^*$$

$$- \frac{(1 + g^2)e^{-gsD}}{\delta J^2} B_2^* = E_{23}$$

$$j^{-1}(1 + k^2)s_ja_2^* + j^{-1}(1 + k^2)c_jb_2^* + 2ks_kc_2^* + 2kc_kd_2^*$$

$$+ \frac{(1 + g^2)e^{-fsD}}{f\delta J^2} A_2^* + \frac{2ge^{-gsD}}{\delta J^2} B_2^* = E_{24}$$

where $E_{19} = - (E_{15} + E_{16})$

$$E_{20} = -\frac{E_{17} + E_{18}}{J^2}$$

$$E_{21} = E_5 e^{-GsD} - E_{11} e^{-FsD} - E_{12} e^{FsD}$$

$$E_{22} = A_3^* e^{-GsD} - A_4^* e^{-FsD} - A_5^* e^{FsD}$$

$$E_{23} = E_7 e^{-GsD} - E_{15} e^{-FsD} - E_{16} e^{FsD}$$

$$E_{24} = E_8 e^{-GsD} - E_{17} e^{-FsD} - E_{18} e^{FsD}$$

or

$$\begin{bmatrix} 2 & 0 & (1+k^2) & 0 & 0 & 0 \\ 0 & j^{-1}(1+k^2) & 0 & 2k & 0 & 0 \\ j^{-1}s_j & j^{-1}c_j & ks_k & kc_k & f^{-1}e^{-fsD} & ge^{-gsD} \\ c_j & s_j & c_k & s_k & -e^{-fsD} & -e^{-gsD} \\ 2c_j & 2s_j & (1+k^2)c_k & (1+k^2)s_k & \frac{-2e^{-fsD}}{\delta J^2} & \frac{-(1+g^2)e^{-gsD}}{\delta J^2} \\ j^{-1}(1+k^2)s_j & j^{-1}(1+k^2)c_j & 2ks_k & 2kc_k & \frac{(1+g^2)e^{-fsD}}{f\delta J^2} & \frac{2ge^{-gsD}}{\delta J^2} \end{bmatrix}$$

$$\begin{bmatrix} a_2^* \\ b_2^* \\ c_2^* \\ d_2^* \\ A_2^* \\ B_2^* \end{bmatrix} = \begin{bmatrix} E_{19} \\ E_{20} \\ E_{21} \\ E_{22} \\ E_{23} \\ E_{24} \end{bmatrix}$$

$$a_2^* = \frac{\Delta_1^*}{\Delta^*}, \quad b_2^* = \frac{\Delta_2^*}{\Delta^*}, \quad c_2^* = \frac{\Delta_3^*}{\Delta^*}, \quad d_2^* = \frac{\Delta_4^*}{\Delta^*}, \quad A_2^* = \frac{\Delta_5^*}{\Delta^*}, \text{ and } B_2^* = \frac{\Delta_6^*}{\Delta^*}$$

where $\Delta^* = \Delta$.

For $s < 0$ we can obtain similar solutions.

The expressions of Δ_1^*, Δ_2^*, Δ_3^*, Δ_4^*, Δ_5^*, and Δ_6^* are given in Appendix II.

CHAPTER III

NUMERICAL SOLUTION

Previous sections have shown the general expressions of the stress and temperature fields in the transformed space. By applying the inverse Fourier transform and following through the simplification of the integrand, we can obtain results for several cases numerically. Here, a discussion of how the thickness of the surface layer affects the thermostress and what material parameters govern conditions leading to crack formation will also be addressed. (Figures 2-13).

3.1 Mechanical Stress Field

$$\sigma_{\xi\xi}^{(1)} = \mathrm{Re}\left\{(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} \tilde{\sigma}_{\xi\xi}^{(1)} e^{-is\xi}\, ds\right\}$$

$$= \mathrm{Re}\left\{\frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} - s\,\frac{\mu_2}{P_o}\,[\delta(M^2 j^{-1} + 2J^2 j)\,(a_1^{*}\sinh j\eta s + b_1^{*}\cosh j\eta s)\right.$$

$$\left. + 2\delta J^2 k(c_1^{*}\sinh k\eta s + d_1^{*}\cosh k\eta s)]e^{-is\xi}\, ds\right\}$$

$$(a_1^{*}\sinh j\eta s + b_1^{*}\cosh j\eta s)\, e^{-is\xi} = \frac{\bar{P}e^{-(f+g)sD}}{\Delta}\left\{\frac{i\mu_f}{\delta J^2}(B_1 + B_2 c_j c_k\right.$$

$$+ B_3 c_j s_k + B_4 s_j s_k + B_5 c_k s_j) - \frac{1}{\delta J^2}(B_6 c_j s_k + B_7 c_j c_k + B_8 c_k s_j$$

$$+ B_9 s_j s_k)]\sinh j\eta s + \left[-\frac{i\mu_f}{\delta J^2}(c_1 s_j c_k + c_2 s_j s_k\right.$$

$$+ c_3 c_j s_k + c_4 c_j c_k) - \frac{1}{\delta J^2}(c_5 + c_6 s_j s_k + c_7 s_j c_k + c_8 c_j c_k$$

$$+ c_9 s_k c_j) \quad \cosh j\eta s \Big\} (\cos s\xi - i \sin s\xi)$$

$$\Big\{ \mathrm{Re}\ (a_1^* \sinh j\eta s + b_1^* \cosh j\eta s) e^{-is\xi} \Big\}$$

$$= \frac{\bar{P}e^{-(f+g)sD}}{\Delta} \Bigg\{ \Bigg[\frac{\mu_f}{\delta J^2} \sin s\xi \ (B_1 + B_2 c_j c_k + B_3 c_j s_k + B_4 s_j s_k$$

$$+ B_5 c_k s_j) - \frac{\cos s\xi}{\delta J^2} (B_6 c_j s_k + B_7 c_j c_k + B_8 c_k s_j$$

$$+ B_9 s_j s_k) \Bigg] \sinh j\eta s + \Bigg[- \frac{\mu_f}{\delta J^2} \sin s\xi \ (c_1 s_j c_k + c_2 s_j s_k$$

$$+ c_3 c_j s_k + c_4 c_j c_k) - \frac{\cos s\xi}{\delta J^2} (c_5 + c_6 s_j s_k + c_7 s_j c_k + c_8 c_j c_k$$

$$+ c_9 s_k c_j) \Bigg] \cosh j\eta s \Bigg\} = \frac{\bar{P}e^{-(f+g)sD}}{\Delta} \Bigg\{ \frac{\mu_f}{\delta J^2} \sin s\xi \ [(B_1 + B_2 c_j c_k$$

$$+ B_3 c_j s_k + B_4 s_j s_k + B_5 c_k s_j) \sinh j\eta s - (c_1 s_j c_k + c_2 s_j s_k$$

$$+ c_3 c_j s_k + c_4 c_j c_k) \cosh j\eta s] - \frac{\cos s\xi}{\delta J^2} [(B_6 c_j s_k + B_7 c_j c_k + B_8 c_k s_j$$

$$+ B_9 s_j s_k) \sinh j\eta s + (c_5 + c_6 s_j s_k + c_7 s_j c_k + c_8 c_j c_k$$

$$+ c_9 s_k c_j) \cosh j\eta s] \Bigg\}$$

$$= \frac{\bar{P}e^{-(f+g)sD}}{\Delta} \Bigg\{ \frac{\mu_f}{\delta J^2} \sin s\xi [B_1 \sinh j\eta s + B_2 c_k (c_j \sinh j\eta s - s_j \cosh j\eta s)$$

$$+ B_3 s_k (c_j \sinh j\eta s - s_j \cosh j\eta s) + B_4 s_k (s_j \sinh j\eta s - c_j \cosh j\eta s)$$

$$+ B_5 c_k (s_j \sinh j\eta s - c_j \cosh j\eta s)] - \frac{\cos s\xi}{\delta J^2} [B_6 s_k (c_j \sinh j\eta s$$

$$- s_j \cosh j\eta s) + B_7 c_k (c_j \sinh j\eta s - s_j \cosh j\eta s) + B_8 c_k (s_j \sinh j\eta s$$

$$- c_j \cosh j\eta s) + B_9 s_k (s_j \sinh j\eta s - c_j \cosh j\eta s) - c_5 \cosh j\eta s] \Bigg\}$$

$$= \frac{\bar{P}e^{-(f+g)sD}}{\Delta}\left\{\frac{\mu_f}{\delta J^2}\ \sin s\xi\ [B_1 \sinh j\eta s - (B_2 c_k + B_3 s_k)\sinh(D - \eta)js\right.$$

$$- (B_4 s_k + B_5 c_k)\cosh(D - \eta)js] + \frac{\cos s\xi}{\delta J^2}\ [(B_6 s_k + B_7 c_k)$$

$$\left.\sinh(D - \eta)js + (B_8 c_k + B_9 s_k)\ \cosh(D - \eta)js - c_5 \cosh j\eta s]\right\}$$

$$(c_1^* \sinh k\eta s + d_1^* \cosh k\eta s)\ e^{-is\xi} = \frac{\bar{P}e^{-(f+g)sD}}{\Delta}\left\{\left[\frac{1}{\delta J^2}\ (d_1 s_k c_j\right.\right.$$

$$+ d_2 c_j c_k + d_3 s_j c_k + d_4 s_j s_k) + \frac{i\mu_f}{\delta J^2}\ (d_5 s_j s_k + d_6 s_j c_k$$

$$\left. + d_7 c_j c_k + d_8 s_k c_j + d_9)\right]\ \sinh k\eta s + \left[\frac{i\mu_f}{\delta J^2}\ (P_1 s_j c_k + P_2 s_j s_k + P_3 s_k c_j\right.$$

$$\left. + P_4 c_j c_k) - \frac{1}{\delta J^2}\ (P_5 c_j c_k + P_6 s_k c_j + P_7 s_j s_k + P_8 s_j c_k + P_9)\ \right]$$

$$\left.\cosh k\eta s\right\}\ (\cos s\xi - i \sin s\xi)$$

$$\mathrm{Re}\left\{(c_1^* \sinh k\eta s + d_1^*\ \cosh k\eta s)\ e^{-is\xi}\right\}$$

$$= \frac{\bar{P}e^{-(f+g)sD}}{\Delta}\left\{\ \frac{\cos s\xi}{\delta J^2}\ [(d_1 s_k c_j + d_2 c_j c_k + d_3 s_j c_k + d_4 s_j s_k)\right.$$

$$\sinh k\eta s - (P_5 c_j c_k + P_6 s_k c_j + P_7 s_j s_k + P_8 s_j c_k + P_9)\cosh k\eta s]$$

$$+ \frac{\mu_f}{\delta J^2}\ \sin s\xi\ [(d_5 s_j s_k + d_6 s_j c_k + d_7 c_j c_k + d_8 s_k c_j + d_9)\sinh k\eta s$$

$$\left. + (P_1 s_j c_k + P_2 s_j s_k + P_3 s_k c_j + P_4 c_j c_k)\cosh k\eta s]\right\}$$

$$= \frac{\bar{P}\ e^{-(f+g)sD}}{\Delta}\left\{\frac{-\cos s\xi}{\delta J^2}\ [(d_1 c_j + d_4 s_j)\cosh(D-\eta)ks\right.$$

$$+ (d_2 c_j + d_3 s_j)\sinh(D - \eta)ks + P_9 \cosh k\eta s] - \frac{\mu_f}{\delta J^2}\ \sin s\xi[(d_5 s_j$$

$$+ d_8 c_j) \cosh(D - \eta)ks + (d_6 s_j + d_7 c_j)\sinh(D - \eta)ks$$

$$- d_9 \sinh k\eta s] \Big\}$$

Therefore
$$\sigma_{\xi\xi}^{(1)} = \frac{2}{\sqrt{2\pi}} \frac{\mu_2}{P_o} \frac{1}{\delta J^2} \int_0^\infty \frac{\bar{P}\, e^{-(f+g)sD}}{\Delta}\ s\, [\delta(M^2 j^{-1} + 2J^2 j)$$

$$\Big\{ \mu_f \sin s\xi\, [- B_1 \sinh j\eta s + (B_2 c_k + B_3 s_k)\sinh(D - \eta)js + (B_4 s_k$$

$$+ B_5 c_k)\cosh(D - \eta)js] - \cos s\xi[(B_6 s_k + B_7 c_k)\sinh(D - \eta)js$$

$$+ (B_8 c_k + B_9 s_k)\cosh(D - \eta)js - c_5 \cosh j\eta s] \Big\} + 2\delta J^2 k \Big\{ \cos s\xi$$

$$[(d_1 c_j + d_4 s_j)\cosh(D - \eta)ks + (d_2 c_j + d_3 s_j)\sinh(D - \eta)ks$$

$$+ P_9 \cosh k\eta s] + \mu_f \sin s\xi\, [(d_5 s_j + d_8 c_j)\cosh(D - \eta)ks$$

$$+ (d_6 s_j + d_7 c_j)\sinh(D - \eta)ks - d_9 \sinh k\eta s] \Big\} \Big]\ ds.$$

Similarly,

$$\sigma_{\xi\eta}^{(1)} = \frac{2}{\sqrt{2\pi}}\ \frac{\mu_2}{P_o} \int_0^\infty \frac{\bar{P}\ e^{-(f+g)sD}}{\Delta}\ s \Big[2 \Big\{ \mu_f \cos s\xi[B_1 \cosh j\eta s + (B_2 c_k + B_3 s_k)$$

$$\cosh(D - \eta)js + (B_4 s_k + B_5 c_k)\sinh(D - \eta)js] + \sin s\xi[(B_6 s_k + B_7 c_k)$$

$$\cosh(D - \eta)js + (B_8 c_k + B_9 s_k)\sinh(D - \eta)js + c_5 \sinh j\eta s] \Big\}$$

$$+ (1 + k^2) \Big\{ - \sin s\xi[(d_1 c_j + d_4 s_j)\sinh(D - \eta)ks$$

$$+ (d_2 c_j + d_3 s_j)\cosh(D - \eta)ks - P_9 \sinh k\eta s] + \mu_f \cos s\xi$$

$$[(d_5 s_j + d_8 c_j)\sinh(D - \eta)ks + (d_6 s_j + d_7 c_j)\cosh(D - \eta)ks$$

$$+ d_9 \cosh k\eta s] \Big\} \Big]\ ds$$

$$\sigma_{\eta\eta}^{(1)} = \frac{2}{\sqrt{2\pi}}\frac{\mu_2}{P_o}\int_0^\infty \frac{\bar{P}\ e^{-(f+g)sD}}{\Delta}\ s\Big[j^{-1}(1+k^2)\Big\{\mu_f \sin s\xi\ [B_1 \sinh j\eta s$$

$$- (B_2c_k + B_3s_k)\sinh(D-\eta)js - (B_4s_k + B_5c_k)\cosh(D-\eta)js]$$

$$+ \cos s\xi[(B_6s_k + B_7c_k)\sinh(D-\eta)js + (B_8c_k + B_9s_k)$$

$$\cosh(D-\eta)js - c_5\cosh j\eta s]\Big\} + 2k\Big\{-\cos s\xi\ [(d_1c_j + d_4s_j)$$

$$\cosh(D-\eta)ks + (d_2c_j + d_3s_j)\ \sinh(D-\eta)ks - P_9\cosh k\eta s]$$

$$- \mu_f\ \sin s\xi\ [(d_5s_j + d_8c_j)\cosh(D-\eta)ks + (d_6s_j + d_7c_j)\ \sinh(D-\eta)ks$$

$$+ d_9\sinh k\eta s]\Big\}\Big]\,ds$$

$$\sigma_{\xi\xi}^{(2)} = \frac{2}{\sqrt{2\pi}}\frac{\mu_2}{P_o}\ \frac{1}{\delta J^2}\int_0^\infty \frac{\bar{P}}{\Delta}\Big\{s\ f^{-1}(M^2 + 2f^2)e^{-(gsD + f\zeta)}$$

$$[\mu_f\sin s\xi(H_1s_j + H_2c_j + H_3c_k + H_4s_k) + \cos s\xi(H_5c_j + H_6s_j$$

$$+ H_7s_k + H_8c_k)] - 2g\ e^{-(fsD+g\zeta)}[\mu_f\sin s\xi(L_1s_j + L_2c_j + L_3c_k$$

$$+ L_4s_k) + \cos s\xi(L_5c_j + L_6s_j + L_7s_k + L_8c_k)]\Big\}\ ds$$

$$\sigma_{\xi\eta}^{(2)} = \frac{2}{\sqrt{2\pi}}\frac{\mu_2}{P_o}\ \frac{1}{\delta J^2}\int_0^\infty \frac{\bar{P}}{\Delta}\ s\Big\{2e^{-(gsD + f\zeta)}\ [\mu_f\cos s\xi\ (H_1s_j + H_2c_j$$

$$+ H_3c_k + H_4s_k) - \sin s\xi(H_5c_j + H_6s_j + H_7s_k + H_8c_k)]$$

$$- (1 + g^2)\ e^{-(fsD+g\zeta)}\ [\mu_f\cos s\xi(L_1s_j + L_2c_j + L_3c_k + L_4s_k)$$

$$- \sin s\xi\ (L_5c_j + L_6s_j + L_7s_k + L_8c_k)]\Big\}\ ds$$

$$\sigma_{nn}^{(2)} = \frac{2}{\sqrt{2\pi}} \frac{\mu_2}{P_o} \frac{1}{\delta J^2} \int_0^\infty \frac{\bar{P}}{\Delta} s \left\{ f^{-1}(1 + g^2) e^{-(gsD+f\zeta)} \right.$$

$$[\mu_f \sin s\xi (H_1 s_j + H_2 c_j + H_3 c_k + H_4 s_k) + \cos s\xi (H_5 c_j + H_6 s_j + H_7 s_k$$

$$+ H_8 c_k)] - 2g e^{-(fsD + g\zeta)} [\mu_f \sin s\xi (L_1 s_j + L_2 c_j + L_3 c_k + L_4 s_k)$$

$$\left. + \cos s\xi \, (L_5 c_j + L_6 s_j + L_7 s_k + L_8 c_k)] \right\} ds$$

The above integrals are too complicated to integrate directly; therefore a numerical integration technique will be employed. There are many numerical schemes for integration. Here subroutine QUANC8 will be used which is based on the 8-panel Newton-Cotes rule [8]. In the integrals, $s = 0$ is a singular point. The method used to avoid the singularity is discussed in Appendix III.

3.2 Temperature Field

$$\overline{\phi^{(1)}} = \frac{\bar{Q}^*}{s} \left[\frac{(F + \beta G)e^{-F\eta s} + (F - \beta G)e^{(\eta - 2D)Fs}}{F^2(1 - e^{-2FsD}) + \beta FG(1 + e^{-2FsD})} \right]$$

Let $F = (1 - iR_1/s)^{1/2} = \gamma_1(\cos\theta_1 + i\sin\theta_1)$

$$G = (1 - iR_2/s)^{1/2} = \gamma_2(\cos\theta_2 + i\sin\theta_2)$$

where

$$\gamma_1 = (1 + R_1^2/s^2)^{1/4}, \quad \theta_1 = 1/2 \tan^{-1}(-R_1/s)$$

$$\gamma_2 = (1 + R_2^2/s^2)^{1/4}, \quad \theta_2 = 1/2 \tan^{-1}(-R_2/s)$$

$$F^2 (1 - e^{-2FsD}) + \beta FG(1 + e^{-2FsD}) = T_1 + iT_2$$

where

$$T_1 = \gamma_1^2 \left\{ \cos 2\theta_1 \left[1 - \cos (2\gamma_1 sD \sin\theta_1) e^{-2\gamma_1 sD\cos\theta_1} \right] \right.$$

$$- \sin 2\theta_1 \sin(2\gamma_1 sD\sin\theta_1)\, e^{-2\gamma_1 sD\cos\theta_1} \Big\} + \beta\gamma_1\gamma_2 \Big\{ \cos(\theta_1 + \theta_2)$$

$$\Big[1 + \cos\,(2\gamma_1 sD\sin\theta_1)\, e^{-2\gamma_1 sD\cos\theta_1}\Big] + \sin(\theta_1 + \theta_2)\sin\,(2\gamma_1 sD\sin\theta_1)$$

$$e^{-2\gamma_1 sD\cos\theta_1}\Big\}$$

$$T_2 = \gamma_1^2 \Big\{ \sin 2\theta_1 \Big[1 - \cos\,(2\gamma_1 sD\sin\theta_1)\, e^{-2\gamma_1 sD\cos\theta_1}\Big] + \cos 2\theta_1$$

$$\sin(2\gamma_1 sD\sin\theta_1) e^{-2\gamma_1 sD\cos\theta_1} \Big\} + \beta\gamma_1\gamma_2 \Big\{ \sin(\theta_1 + \theta_2)\ [1 + \cos(2\gamma_1 sD\sin\theta_1)$$

$$e^{-2\gamma_1 sD\cos\theta_1}] - \cos(\theta_1 + \theta_2)\ \sin(2\gamma_1 sD\sin\theta_1)\ e^{-2\gamma_1 sD\cos\theta_1}\Big\}$$

$$(F + \beta G)e^{-F\eta s} = T_3 + iT_4$$

where

$$T_3 = e^{-\gamma_1 s\eta\cos\theta_1}\ [\cos(\gamma_1 s\eta\sin\theta_1)(\gamma_1\cos\theta_1 + \beta\gamma_2\cos\theta_2) + \sin(\gamma_1 s\eta\sin\theta_1)$$

$$(\gamma_1\sin\theta_1 + \beta\gamma_2\sin\theta_2)]$$

$$T_4 = e^{-\gamma_1 s\eta\cos\theta_1}\ [\cos(\gamma_1 s\eta\sin\theta_1)(\gamma_1\sin\theta_1 + \beta\gamma_2\sin\theta_2) - \sin(\gamma_1 s\eta\sin\theta_1)$$

$$(\gamma_1\cos\theta_1 + \beta\gamma_2\cos\theta_2)]$$

$$(F - \beta G\ )\ e^{(\eta-2D)Fs} = T_5 + iT_6$$

where

$$T_5 = e^{(\eta-2D)s\gamma_1\cos\theta_1} \Big\{ \cos[(\eta - 2D)s\gamma_1\sin\theta_1](\gamma_1\cos\theta_1 - \beta\gamma_2\cos\theta_2)$$

$$- \sin[(\eta - 2D)s\gamma_1\sin\theta_1](\gamma_1\sin\theta_1 - \beta\gamma_2\sin\theta_2)\Big\}$$

$$T_6 = e^{(\eta-2D)s\gamma_1\cos\theta_1} \Big\{ \cos[(\eta - 2D)s\gamma_1\sin\theta_1](\gamma_1\sin\theta_1 - \beta\gamma_2\sin\theta_2)$$

$$+ \sin[(\eta - 2D)s\gamma_1 \sin\theta_1](\gamma_1\cos\theta_1 - \beta\gamma_2\cos\theta_2)\Big\}$$

$$\overline{\phi^{(1)}} = \frac{\bar{Q}^*}{s}\left[\frac{(T_3 + iT_4) + (T_5 + iT_6)}{T_1 + iT_2}\right] = \frac{\bar{Q}^*}{s}(T_7 + iT_8)$$

where

$$T_7 = \frac{T_1(T_3 + T_5) + T_2(T_4 + T_6)}{T_1^{\,2} + T_2^{\,2}}$$

$$T_8 = \frac{T_1(T_4 + T_6) - T_2(T_3 + T_5)}{T_1^{\,2} + T_2^{\,2}}$$

$$\phi^{(1)} = \mathrm{Re}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{\phi}^{(1)}\, e^{-is\xi}\, ds\right\}$$

$$= \mathrm{Re}\left\{\frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} \bar{Q}^*\,(T_7 + iT_8)\, e^{-is\xi}\, ds\right\}$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} \bar{Q}^*\,(T_7\cos s\xi + T_8\sin s\xi)\, ds.$$

Similarly,

$$\phi^{(2)} = \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} \bar{Q}^*\,(T_{11}\cos s\xi + T_{12}\sin s\xi)\, ds$$

where

$$T_{11} = \frac{T_1T_9 + T_2T_{10}}{T_1^{\,2} + T_2^{\,2}}$$

$$T_{12} = \frac{T_1T_{10} - T_2T_9}{T_1^{\,2} + T_2^{\,2}}$$

$$T_9 = 2\gamma_1 e^{[-\gamma_1 sD\cos\theta_1 + (D-\eta)s\gamma_2\cos\theta_2]} \Big\{ \cos\theta_1 \cos [-\gamma_1 sD\sin\theta_1$$

$$+ (D - \eta)s\gamma_2\sin\theta_2] - \sin\theta_1 \sin [-r_1 sD\sin\theta_1 + (D - \eta)s\gamma_2\sin\theta_2] \Big\}$$

$$T_{10} = 2\gamma_1\, e^{[-\gamma_1 sD\cos\theta_1 + (D-\eta)s\gamma_2\cos\theta_2]} \Big\{ \cos\theta_1 \sin [-\gamma_1 sD\sin\theta_1$$

$$+ (D - \eta)s\gamma_2\sin\theta_2] + \sin\theta_1 \cos [-\gamma_1 sD\mathrm{sins}\theta_1 + (D - \eta)s\gamma_2\sin\theta_2] \Big\}$$

3.3 Thermal Stress Field

$$\sigma^{(1)}_{\xi\xi} = \frac{2}{\sqrt{2\pi}} \frac{\mu_2}{P_o} \delta \int_0^\infty \frac{\bar{Q}^*}{\Delta} F_1 \, ds$$

where

$$F_1 = -(M^2 j^{-1} + 2J^2 j) \Big[\cos s\xi \Big\{ -E^\gamma_{19} [-A_1 \sinh j\eta s + (A_3 c_k + A_5 s_k)$$

$$\sinh(D - \eta)js + (A_7 s_k + A_9 c_k)\cosh(D - \eta)js] - (1 + k^2)$$

$$[(B^\gamma_6 s_k + B^\gamma_7 c_k) \sinh(D - \eta)js + (B^\gamma_8 c_k + B^\gamma_9 s_k)\cosh(D - \eta)js$$

$$- (B^\gamma_{10} s_k + B^\gamma_{11} c_k + B^\gamma_{12} sj + B^\gamma_{13} c_j) \sinh j\eta s + (B^\gamma_{12} c_j + B^\gamma_{13} s_j)$$

$$\cosh j\eta s] + 2(c^\gamma_1 + c^\gamma_2 c_k + c^\gamma_3 s_k)\cosh j\eta s \Big\} + \sin s\xi \Big\{ - E^i_{19}$$

$$[- A_1 \sinh j\eta s + (A_3 c_k + A_5 s_k)\sinh(D - \eta)js + (A_7 s_k + A_9 c_k)$$

$$\cosh(D - \eta)js] - (1 + k^2)[(B^i_6 s_k + B^i_7 c_k)\sinh(D - \eta)js + (B^i_8 c_k + B^i_9 s_k)$$

$$\cosh(D - \eta)js - (B^i_{10} s_k + B^i_{11} c_k + B^i_{12} s_j + B^i_{13} c_j) \sinh j\eta s$$

$$+ (B^i_{12} c_j + B^i_{13}) \cosh j\eta s] + 2(c^i_1 + c^i_2 c_k + c^i_3 s_k) \cosh j\eta s \Big\} \Big]$$

$$+ 2J^2k\,[\cos s\xi \Big\{ 2[(B^{\gamma}_{10}s_k + B^{\gamma}_{11}c_k)\sinh ks\eta + (d^{\gamma}_3 c_j + d^{\gamma}_6 s_j)\cosh(D - \eta)ks$$

$$+ (d^{\gamma}_4 c_j + d^{\gamma}_5 s_j)\sinh(D - \eta)ks + (B^{\gamma}_{12}s_j + B^{\gamma}_{13}c_j)\sinh k\eta s + (B^{\gamma}_{10}c_k$$

$$+ B^{\gamma}_{11}s_k)\cosh k\eta s] + E^{\gamma}_{19}\,[(A_8 s_j + A_6 c_j)\cosh(D - \eta)ks + (A_{10}s_j + A_4 c_j)$$

$$\sinh(D - \eta)ks - A_2\sinh ks\eta] + (1 + k^2)(P^{\gamma}_7 c_j + P^{\gamma}_8 s_j + P^{\gamma}_9)\cosh k\eta s \Big\}$$

$$+ \sin s\xi \Big\{ 2[(B^{i}_{10}s_k + B^{i}_{11}c_k)\sinh k\eta s + (d^{i}_3 c_j + d^{i}_6 s_j)\cosh(D - \eta)ks$$

$$+ (d^{i}_4 c_j + d^{i}_5 s_j)\sinh(D - \eta)ks + (B^{i}_{12}s_j + B^{i}_{13}c_j)\sinh k\eta s + (B^{i}_{10}c_k$$

$$+ B^{i}_{11}s_k)\cosh k\eta s] + E^{i}_{19}\,[(A_8 s_j + A_6 c_j)\cosh(D - \eta)ks + (A_{10}s_j + A_4 c_j)$$

$$\sinh(D - \eta)ks - A_2\sinh k\eta s] + (1 + k^2)(P^{i}_7 c_j + P^{i}_8 s_j + P^{i}_9)\cosh k\eta s \Big\}]$$

$$+ T_{13}\cos s\xi - T_{14}\sin s\xi$$

where the superscript γ designates the real part of that constant, i the imaginary part. For example, $E_{20} = E^{\gamma}_{20} + iE^{i}_{20}$.

$$T_{13} = e^{-\gamma_1 s\eta\cos\theta_1}\,[E^{\gamma}_{13}\cos(r_1 s\eta\sin\theta_1) + E^{i}_{13}\sin(\gamma_1 s\eta\sin\theta_1)]$$

$$+ e^{\gamma_1 s\eta\cos\theta_1}\,[E^{\gamma}_{14}\cos(\gamma_1 s\eta\sin\theta_1) - E^{i}_{14}\sin(\gamma_1 s\eta\sin\theta_1)]$$

$$T_{14} = e^{-\gamma_1 s\eta\cos\theta_1}[E^{i}_{13}\cos(\gamma_1 s\eta\sin\theta_1) - E^{\gamma}_{13}\sin(\gamma_1 s\eta\sin\theta_1)]$$

$$+ e^{\gamma_1 s\eta\cos\theta_1}\,[E^{i}_{14}\cos(\gamma_1 s\eta\sin\theta_1) + E^{\gamma}_{14}\sin(\gamma_1 s\eta\sin\theta_1)]$$

$$\sigma^{(1)}_{\xi\eta} = \frac{2}{\sqrt{2\pi}}\frac{\mu_2}{P_0}\,\delta J^2\int_0^{\infty} -\frac{\bar{Q}^*}{\Delta}F_2\,ds$$

where

$$
\begin{aligned}
F_2 = {} & 2[\sin s\xi \Big\{ E_{19}^{\gamma}[(A_3 c_k + A_5 s_k)\cosh(D - \eta)js + (A_7 s_k + A_9 c_k) \\
& \sinh(D - \eta)js + A_1 \cosh j\eta s] + (1 + k^2)[B_6{}^{\gamma} s_k + B_7{}^{\gamma} c_k) \\
& \cosh(D - \eta)js + (B_8^{\gamma} c_k + B_9^{\gamma} s_k)\sinh(D - \eta)js + (B_{10}^{\gamma} s_k + B_{11}^{\gamma} c_k \\
& + B_{12}^{\gamma} s_j + B_{13}^{\gamma} c_j)\cosh j\eta s - (B_{12}^{\gamma} c_j + B_{13}^{\gamma} s_j)\sinh j\eta s] + 2(c_1^{\gamma} + c_2^{\gamma} c_k \\
& + c_3^{\gamma} s_k)\sinh j\eta s \Big\} - \cos s\xi \Big\{ E_{19}^{i}[(A_3 c_k + A_5 s_k)\cosh(D - \eta)js \\
& + (A_7 s_k + A_9 c_k)\sinh(D - \eta)js + A_1 \cosh j\eta s] + (1 + k^2)[B_6^{i}\, s_k \\
& + B_7^{i}\, c_k)\cosh(D - \eta)js + (B_8^{i}\, c_k + B_9^{i}\, s_k)\sinh(D - \eta)js + (B_{10}^{i}\, s_k \\
& + B_{11}^{i}\, c_k + B_{12}^{i}\, s_j + B_{13}^{i}\, c_j)\cosh j\eta s - (B_{12}^{i}\, c_j + B_{13}^{i}\, s_j)\sinh j\eta s] \\
& + 2(c_1^{i} + c_2^{i}\, c_k + c_3^{i}\, s_k)\sinh j\eta s \Big\} \Big] + (1 + k^2)\ [\sin s\xi \\
& \Big\{ 2[(d_3^{\gamma}\, c_j + d_6^{\gamma}\, s_j)\sinh(D - \eta)ks + (d_4^{\gamma}\, c_j + d_5^{\gamma}\, s_j)\cosh(D - \eta)ks \\
& - (B_{10}^{\gamma}\, s_k + B_{11}^{\gamma}\, c_k + B_{12}^{\gamma}\, s_j + B_{13}^{\gamma}\, c_j)\cosh k\eta s + (B_{10}^{\gamma}\, c_k + B_{11}^{\gamma}\, s_k) \\
& \sinh k\eta s] + E_{19}^{\gamma}\ [(A_8 s_j + A_6 c_j)\sinh(D - \eta)ks + (A_{10} s_j + A_4 c_j) \\
& \cosh(D - \eta)ks + A_2 \cosh k\eta s] + (1 + k^2)(P_7^{\gamma}\, c_j + P_8^{\gamma}\, s_j + P_9^{\gamma}) \\
& \sinh k\eta s \Big\} - \cos s\xi \Big\{ 2[(d_3^{i}\, c_j + d_6^{i}\, s_j)\sinh(D - \eta)ks + (d_4^{i}\, c_j + d_5^{i}\, s_j) \\
& \cosh(D - \eta)ks - (B_{10}^{i}\, s_k + B_{11}^{i}\, c_k + B_{12}^{i}\, s_j + B_{13}^{i}\, c_j)\cosh k\eta s \\
& + (B_{10}^{i}\, c_k + B_{11}^{i}\, s_k)\sinh k\eta s] + E_{19}^{i}\ [(A_8 s_j + A_6 c_j)\sinh(D - \eta)ks
\end{aligned}
$$

$$+ (A_{10}s_j + A_4c_j)\cosh(D - \eta)ks + A_2\cosh k\eta s] + (1 + k^2)(P_7^i\ c_j$$

$$+ P_8^i\ s_j + P_9^i\ )\sinh k\eta s\}\Big] + T_{15}\sin s\xi - T_{16}\cos s\xi$$

$$T_{15} = e^{-\gamma_1 s\eta\cos\theta_1}[E_{15}^{\gamma}\ \cos(\gamma_1 s\eta\sin\theta_1) + E_{15}^{i}\ \sin(\gamma_1 s\eta\sin\theta_1)]$$

$$+ e^{\gamma_1 s\eta\cos\theta_1}\ [E_{16}{}^{\gamma}\cos(\gamma_1 s\eta\sin\theta_1) - E_{16}{}^{i}\sin(\gamma_1 s\eta\sin\theta_1)]$$

$$T_{16} = e^{-\gamma_1 s\eta\cos\theta_1}[E_{15}^{i}\ \cos(\gamma_1 s\eta\sin\theta_1) - E_{15}^{\gamma}\ \sin(r_1 s\eta\sin\theta_1)]$$

$$+ e^{\gamma_1 s\eta\cos\theta_1}[E_{16}{}^{i}\cos(\gamma_1 s\eta\sin\theta_1) + E_{16}^{\gamma}\ \sin(\gamma_1 s\eta\sin\theta_1)]$$

$$\sigma_{\eta\eta}^{(1)} = \frac{2}{\sqrt{2\pi}}\ \frac{\mu_2}{P_0}\ \delta\int_0^\infty \frac{\bar{Q}^*}{\Delta}\ F_3\ ds$$

where

$$F_3 = \frac{J^2(1 + k^2)}{j}\ [\cos s\xi\ \Big\{- E_{19}{}^{\gamma}[- A_1\sinh j\eta s + (A_3c_k + A_5s_k)$$

$$\sinh(D - \eta)js + (A_7s_k + A_9c_k)\cosh(D - \eta)js] - (1 + k^2)$$

$$[(B_6^{\gamma}\ s_k + B_7^{\gamma}\ c_k)\sinh(D - \eta)js + (B_8^{\gamma}\ c_k + B_9^{\gamma}\ s_k)\ \cosh(D - \eta)js$$

$$- (B_{10}^{\gamma}\ s_k + B_{11}^{\gamma}\ c_k + B_{12}^{\gamma}\ s_j + B_{13}^{\gamma}\ c_j)\sinh j\eta s + (B_{12}^{\gamma}\ c_j + B_{13}^{\gamma}\ s_j)$$

$$\cosh j\eta s] + 2(c_1{}^{\gamma} + c_2{}^{\gamma}c_k + c_3{}^{\gamma}s_k)\cosh j\eta s\Big\} + \sin s\xi\ \Big\{- E_{19}{}^{i}$$

$$[- A_1\sinh j\eta s + (A_3c_k + A_5s_k)\sinh(D - \eta)js' + (A_7s_k + A_9c_k)$$

$$\cosh(D - \eta)js] - (1 + k^2)[(B_6^{i}\ s_k + B_7^{i}\ c_k)\sinh(D - \eta)js$$

$$+ (B_8^{i}\ c_k + B_9^{i}\ s_k)\cosh(D - \eta)js - (B_{10}^{i}\ s_k + B_{11}^{i}\ c_k + B_{12}^{i}\ s_j + B_{13}^{i}c_j)$$

$$\sinh j\eta s + (B_{12}^{i} c_j + B_{13}^{i} s_j) \cosh j\eta s] + 2(c_1^{i} + c_2^{i} c_k + c_3^{i} s_k)$$

$$\cosh j\eta s\Big\}] + 2kJ^2 [\cos s\xi \Big\{ - 2[(B_{10}^{\gamma} s_k + B_{11}^{\gamma} c_k)\sinh k\eta s$$

$$+ (d_3^{\gamma} c_j + d_6^{\gamma} s_j)\cosh(D - \eta)ks + (d_4^{\gamma} c_j + d_5^{\gamma} s_j)\sinh(D - \eta)ks - (B_{10}^{\gamma} c_k$$

$$+ B_{11}^{\gamma} s_k)\cosh k\eta s + (B_{12}^{\gamma} s_j + B_{13}^{\gamma} c_j)\sinh k\eta s] - E_{19}^{\gamma}[(A_8 s_j + A_6 c_j)$$

$$\cosh(D - \eta)ks + (A_{10} s_j + A_4 c_j)\sinh(D - \eta)ks - A_2 \sinh k\eta s]$$

$$+ (1 + k^2)(P_7^{\gamma} c_j + P_8^{\gamma} s_j + P_9^{\gamma})\cosh k\eta s\Big\} + \sin s\xi \Big\{ - 2[(B_{10}^{i} s_k$$

$$+ B_{11}^{i} c_k)\sinh k\eta s + (d_3^{i} c_j + d_6^{i} s_j)\cosh(D - \eta)ks + (d_4^{i} c_j + d_5^{i} s_j)$$

$$\sinh(D - \eta)ks - (B_{10}^{i} c_k + B_{11}^{i} s_k)\cosh k\eta s + (B_{12}^{i} s_j + B_{13}^{i} c_j)$$

$$\sinh k\eta s] - E_{19}^{i} [(A_8 s_j + A_6 c_j)\cosh(D - \eta)ks + (A_{10} s_j + A_4 c_j)$$

$$\sinh(D - \eta)ks - A_2 \sinh k\eta s] + (1 + k^2)(P_7^{i} c_j + P_8^{i} s_j + P_9^{i})$$

$$\cosh k\eta s\Big\}] + T_{17} \cos s\xi - T_{18} \sin s\xi$$

$$T_{17} = e^{-\gamma_1 s\eta \cos\theta_1} [E_{17}^{\gamma} \cos(\gamma_1 s\eta \sin\theta_1) + E_{17}^{i} \sin(\gamma_1 s\eta \sin\theta_1)]$$

$$+ e^{\gamma_1 s\eta \cos\theta_1} [E_{18}^{\gamma} \cos(\gamma_1 s\eta \sin\theta_1) - E_{18}^{i} \sin(\gamma_1 s\eta \sin\theta_1)]$$

$$T_{18} = e^{-\gamma_1 s\eta \cos\theta_1} [E_{17}^{i} \cos(\gamma_1 s\eta \sin\theta_1) - E_{17}^{\gamma} \sin(\gamma_1 s\eta \sin\theta_1)]$$

$$+ e^{\gamma_1 s\eta\cos\theta_1}[E_{18}^{i}\cos(\gamma_1 s\eta\sin\theta_1) + E_{18}^{\gamma}\sin(\gamma_1 s\eta\sin\theta_1)]$$

$$\sigma_{\xi\xi}^{(2)} = \frac{2}{\sqrt{2\pi}}\frac{\mu_2}{P_o}\int_0^\infty \frac{s}{\Delta}\Big\{f^{-1}(M^2 + 2f^2)[(\Delta_5^*)^\gamma\cos s\xi + (\Delta_5^*)^i\sin s\xi]e^{-f\eta s}$$

$$+ 2g[(\Delta_6^*)^\gamma\cos s\xi + (\Delta_6^*)^i\sin s\xi]\,e^{-g\eta s} + (T_{19}\cos s\xi + T_{20}\sin s\xi)\Big\}ds$$

where

$$T_{19} = e^{-\gamma_2 s\eta\cos\theta_2}[E_6^\gamma\cos(\gamma_2 s\eta\sin\theta_2) + E_6^i\sin(\gamma_2 s\eta\sin\theta_2)]$$

$$T_{20} = e^{-\gamma_2 s\eta\cos\theta_2}[E_6^i\cos(\gamma_2 s\eta\sin\theta_2) - E_6^\gamma\sin(\gamma_2 s\eta\sin\theta_2)]$$

$$\sigma_{\xi\eta}^{(2)} = \frac{2}{\sqrt{2\pi}}\frac{\mu_2}{P_o}\int_0^\infty -\frac{s}{\Delta}\Big\{2[(\Delta_5^*)^\gamma\sin s\xi - (\Delta_5^*)^i\cos s\xi]e^{-f\eta s}$$

$$+ (1 + g^2)[(\Delta_6^*)^\gamma\sin s\xi - (\Delta_6^*)^i\cos s\xi]\,e^{-g\eta s} + (T_{21}\sin s\xi$$

$$-T_{22}\cos s\xi)\Big\}\,ds$$

where

$$T_{21} = e^{-\gamma_2 s\eta\cos\theta_2}[E_7^\gamma\cos(\gamma_2 s\eta\sin\theta_2) + E_7^i\sin(\gamma_2 s\eta\sin\theta_2)]$$

$$T_{22} = e^{-\gamma_2 s\eta\cos\theta_2}[E_7^i\cos(\gamma_2 s\eta\sin\theta_2) - E_7^\gamma\sin(\gamma_2 s\eta\sin\theta_2)]$$

$$\sigma_{\eta\eta}^{(2)} = \frac{2}{\sqrt{2\pi}}\frac{\mu_2}{P_o}\int_0^\infty \frac{-s}{\Delta}\Big\{\frac{(1 + g^2)}{f}[(\Delta_5^*)^\gamma\cos s\xi + (\Delta_5^*)^i\sin s\xi]\,e^{-f\eta s}$$

$$+ 2g[(\Delta_6^*)^\gamma\cos s\xi + (\Delta_6^*)^i\sin s\xi]\,e^{-g\eta s} + (T_{23}\cos s\xi + T_{24}\sin s\xi)\Big\}ds$$

where

$$T_{23} = e^{-\gamma_2 s\eta\cos\theta_2} [E_8^{\gamma} \cos(\gamma_2 s\eta\cos\theta_2) + E_8^{i} \sin(\gamma_2 s\eta\sin\theta_2)]$$

$$T_{24} = e^{-\gamma_2 s\eta\cos\theta_2} [E_8^{i} \cos(\gamma_2 s\eta\sin\theta_2) - E_8^{\gamma} \sin(\gamma_2 s\eta\sin\theta_2)]$$

CHAPTER IV

CONCLUSION

In the rapid traversing friction source model, the thermal effect dominates the stress field that eventually leads to failure. Numerical results are presented in the text corresponding to five different cases of material properties and geometry to study the layer thickness effect, the stiffness ratio effect, the thermal diffusivity ratio effect, and the effect of pressure distribution. Only the stresses at $\xi > 0$ are discussed, since the maximum stress occurs at $0 < \xi \leq 1$ in all the cases. Case 1 (Figures 2-4) illustrates the effect of a soft layer for which the material of the surface layer is represented by zirconium; the material of the substrate is stellite III which is harder than zirconium; the dimensionless layer thickness D is two; and the pressure distribution is uniform. Figure 4 shows that the maximum thermal stress for case 1 is 1.1 p_0 . Case 2 (Figures 5-6) illustrates the thickness effect, for which the materials of the surface layer and the substrate are the same as in case 1, but D is different. Figure 6 shows that the maximum thermal stresses are 1.25 p_0 , 1.6 p_0 , and 1.05 p_0 for D= 0.01, 0.1, and 2.0, respectively. The phenomenon indicates that the stress is maximum when the layer thickness is in the neighborhood of $\eta = 0.1$, the depth of maximum thermal gradient. Case 3 (Figures 7-8) illustrates the effect of thermal diffusivity ratio, in which the material properties are the same as those in case 1 except that the thermal diffusivity of the layer material is one-half that of zirconium, with D=2. In this case, both the temperature and the thermal stress decrease in comparison to those of case 1. In case 4 (Figures 9-10) the effect of a hard coating is

illustrated, in which the material properties are the same as those in case 1 except that the Young's modulus of the layer material is five times that of zirconium, with D= 2. It is shown in Figure 10 that the thermal stress is much larger than that in case 1. In case 5 the effect of pressure profile is studied. In this case (Figures 11-13) the pressure distribution is parabolic and the total pressure is the same as the uniform pressure case and the material properties are the same as in case 1. The thermal stress due to this type of excitation is larger than that with uniformly distributed pressure.

Base on the cases studied, it may thus be concluded that a stiff surface layer, which is less compliant, would result in higher thermal stress. Surface layer materials with low thermal diffusivity are generally a result of high thermal capacity which in turn, results in less temperature rise, thus lower thermal stress. Finally, the effect of surface layer thickness depends on the thermal layer which is defined by the depth of maximum temperature gradient. Therefore, substantial reduction in thermal stress can be achieved by decreasing the thermal diffusivity, by decreasing the modulus of elasticity of the surface layer, or by designing the layer thickness such that the interface is away from the depth of maximum temperature gradient, which in the present numerical problem is η =0.1.

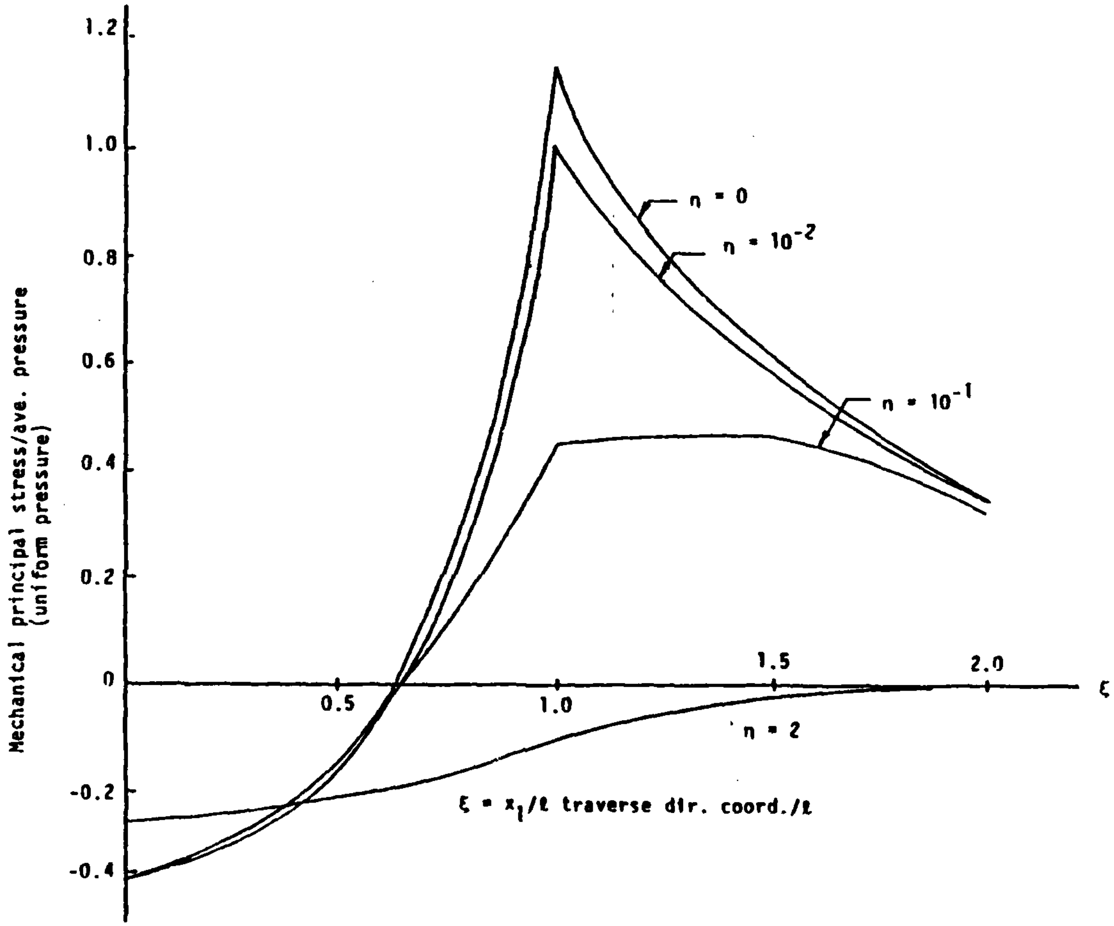

Figure 2. Dimensionless Mechanical Principal Stress

The material of the surface layer is zirconium. The material of the substrate is stellite III. $D = H/\ell = 2$, $\ell = 0.01$ in.

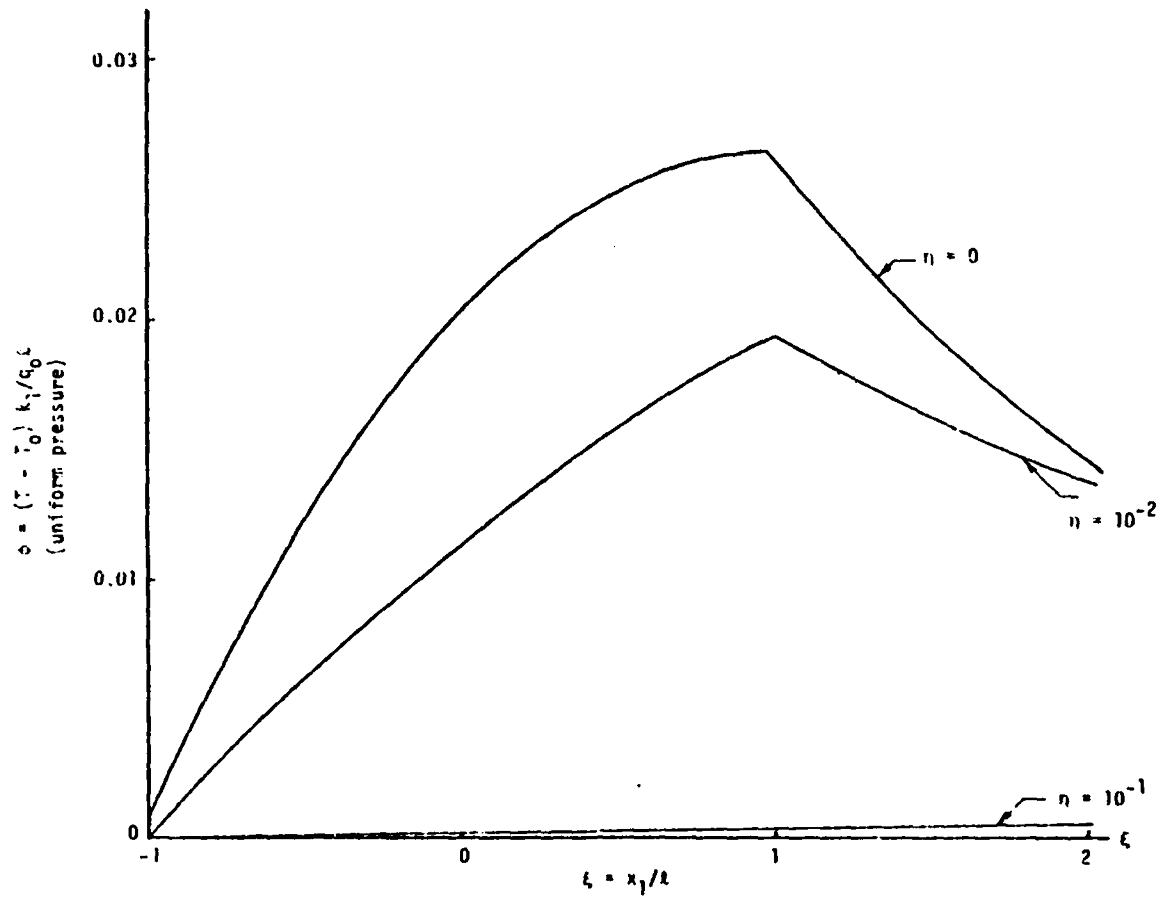

Figure 3. Dimensionless Temperature

The materials of the surface layer and the substrate are the same as Figure 2. $D = H/\ell = 2$, $\ell = 0.01$ in.

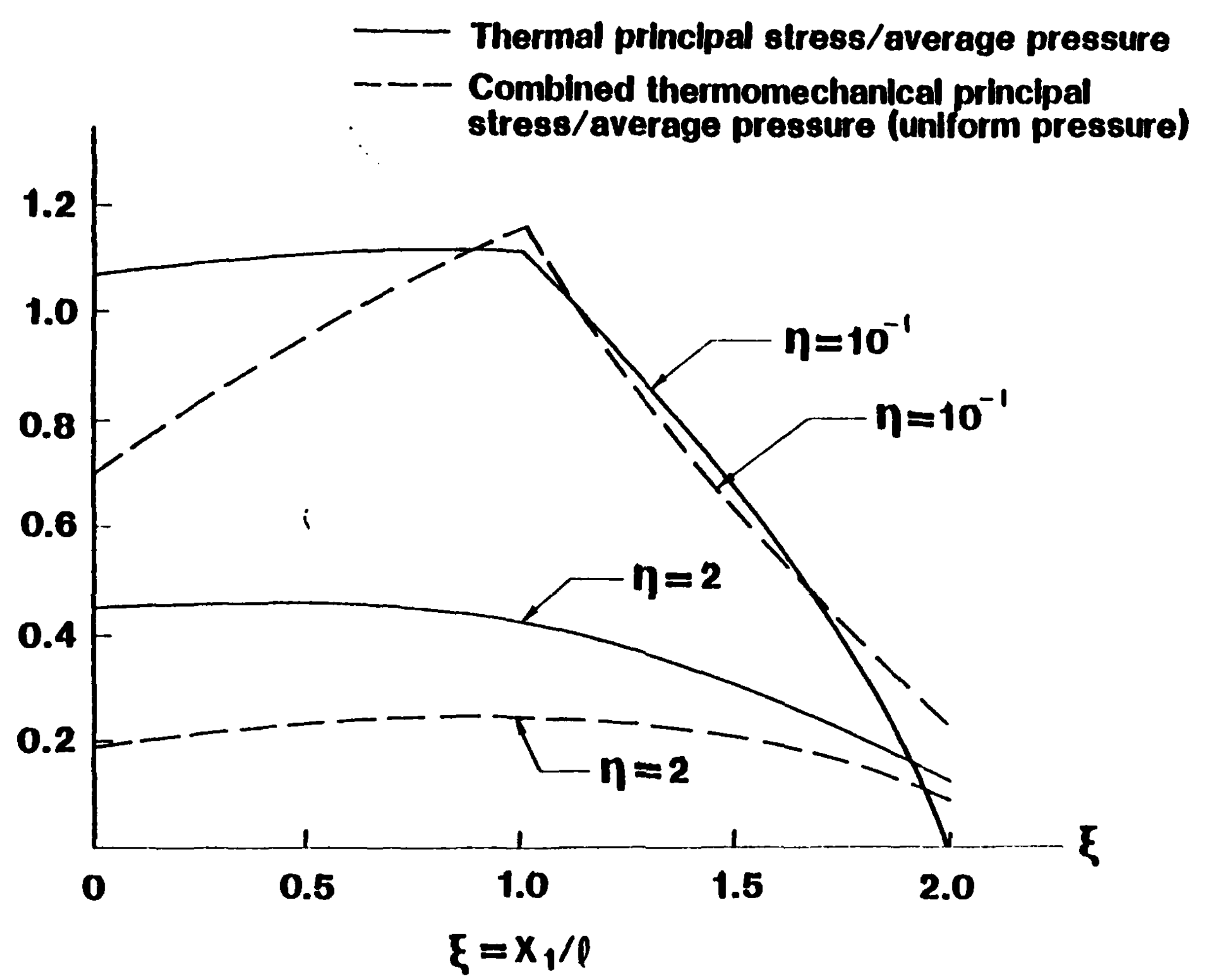

Figure 4. Dimensionless Thermal and Combined Thermomechanical Principal Stresses

The materials of the surface layer and the substrate are the same as Figure 1. D=2, l=0.01 in.

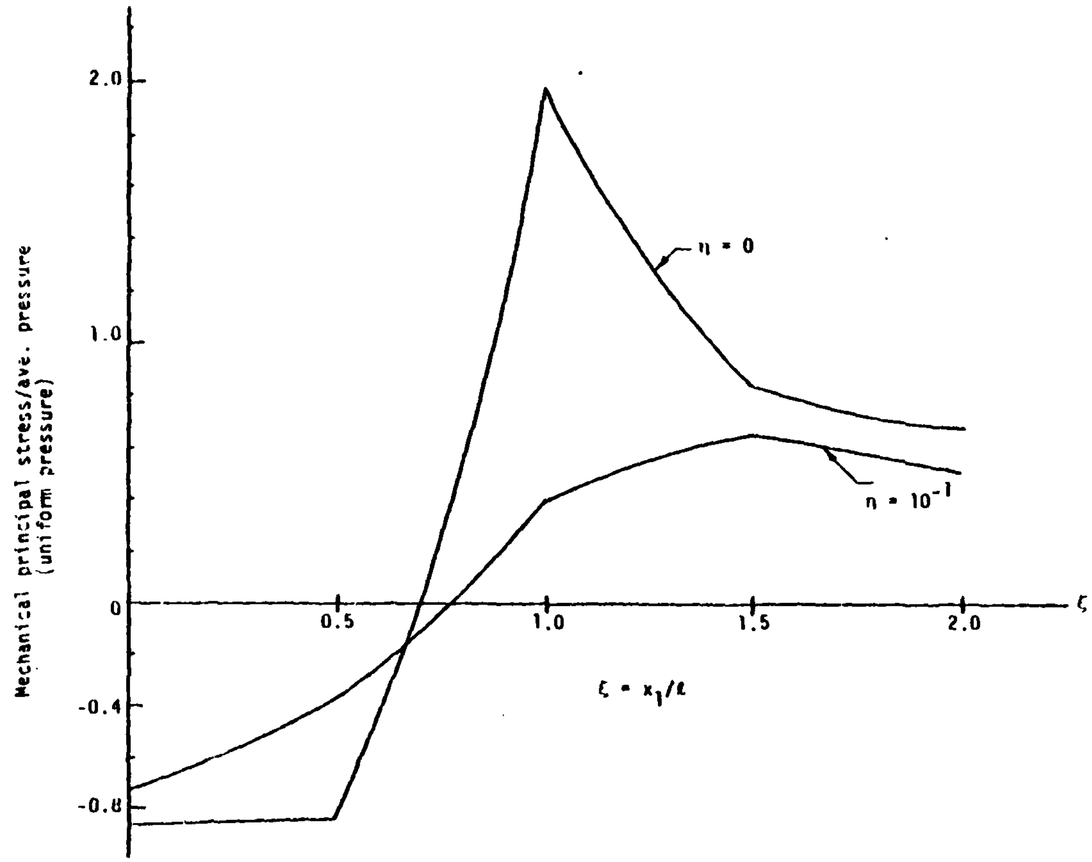

Figure 5. Dimensionless Mechanical Principal Stress

The materials of the surface layer and the substrate are the same as Figure 2. $D = H/\ell = 0.1$, $\ell = 0.01$ in.

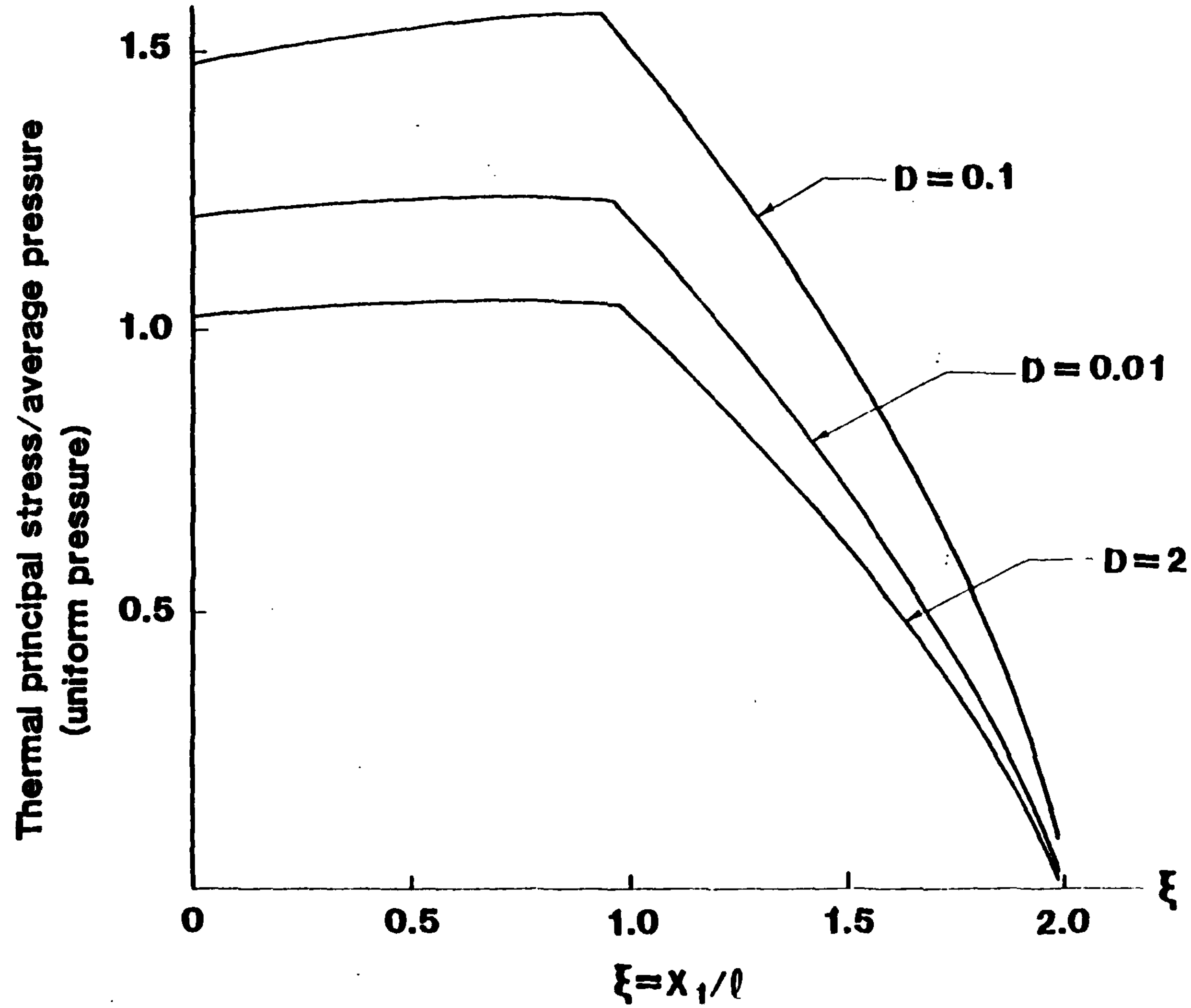

Figure 6. Dimensionless Thermal Principal Stresses for Different values of D

The materials of the surface layer and the substrate are the same as Figure 2. η=0.1, 1=0.01 in.

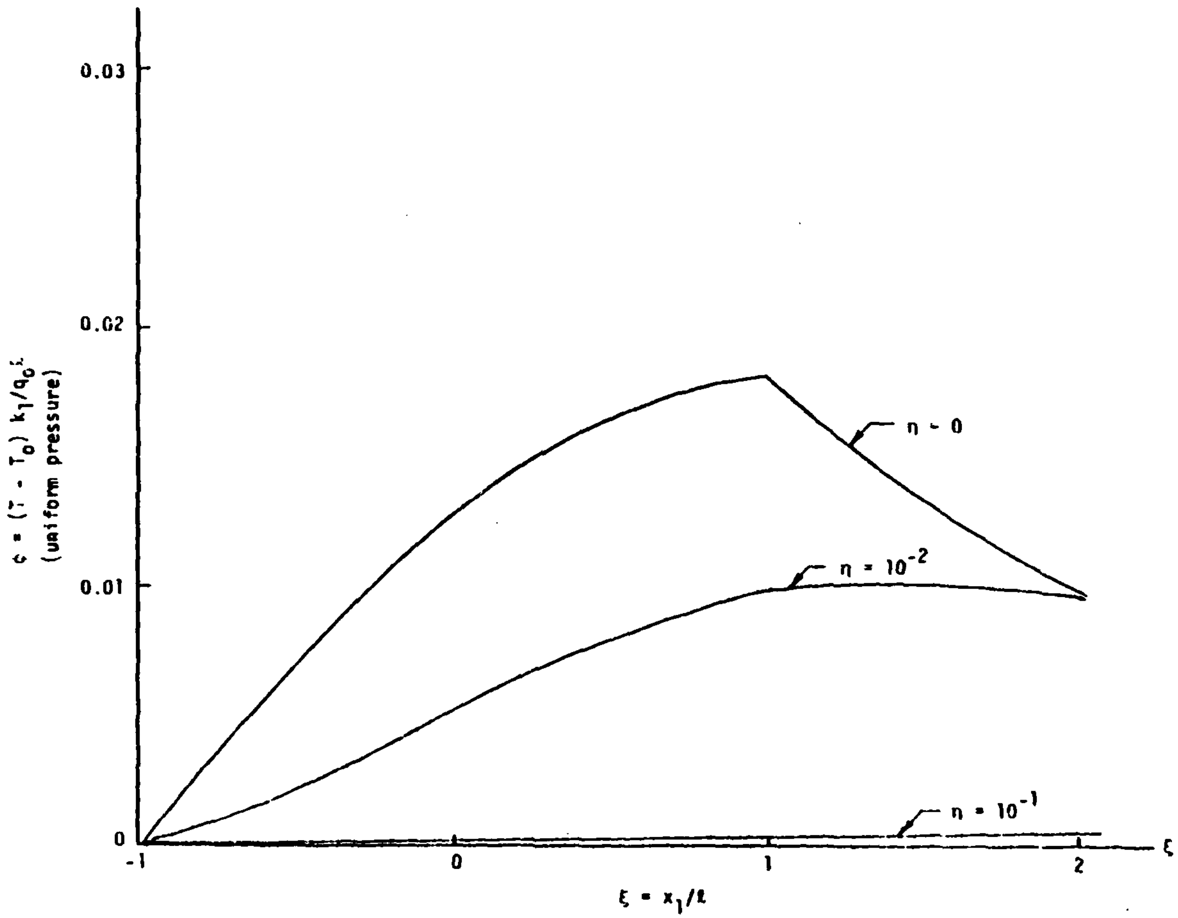

Figure 7. Dimensionless Temperature

The material properties of the surface layer are the same as zirconium excent thermal diffusivity is 1/2 that of zirconium. The material of the substrate is stellite III. $D = H/\ell = 2$, $\ell = 0.01$ in.

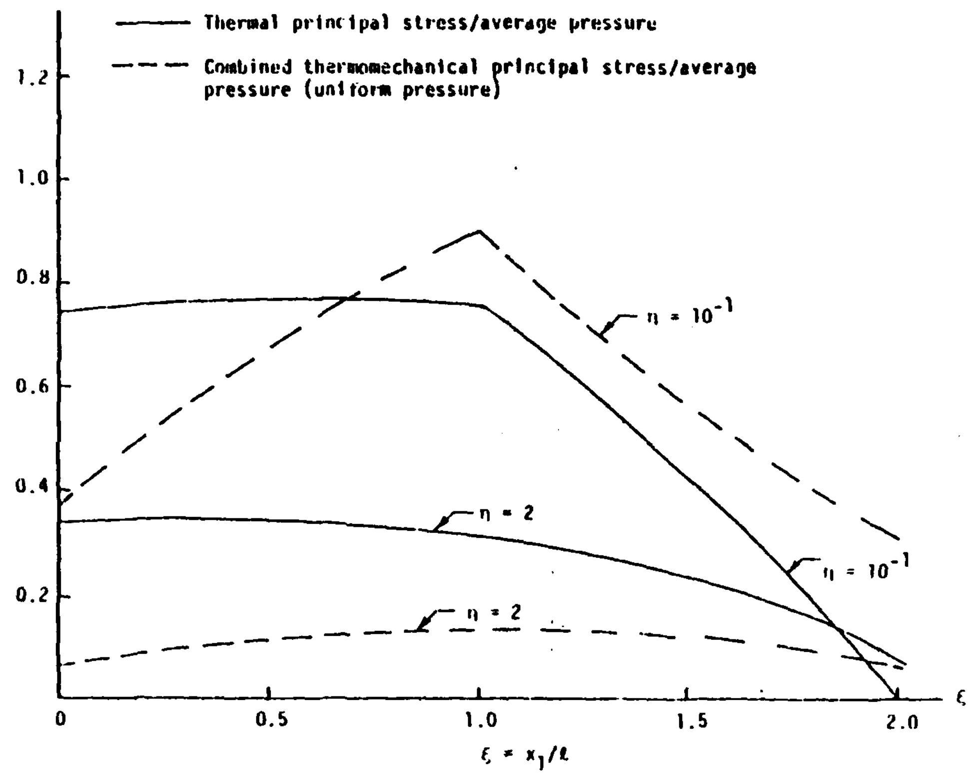

Figure 8. Dimensionless Thermal and Combined Thermomechanical Principal Stresses

The materials of the surface layer and the substrate are the same as Figure 7. $D = H/\ell = 2$, $\ell = 0.01$ in.

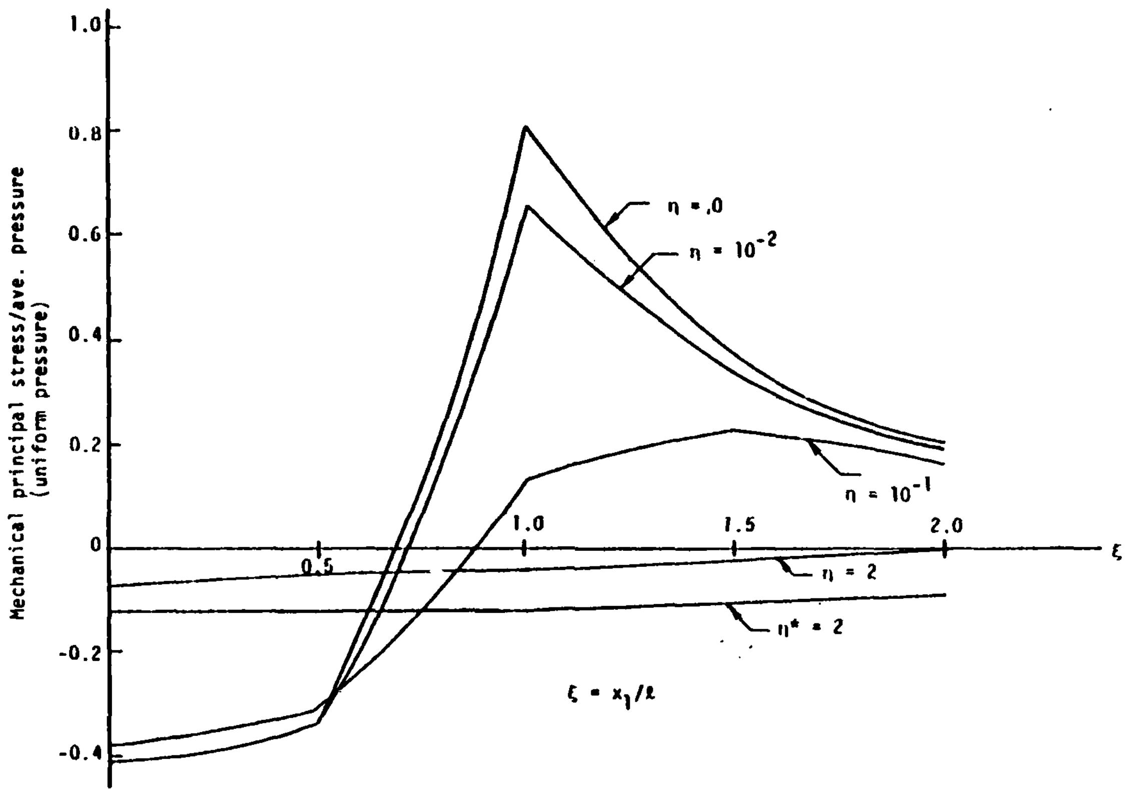

Figure 9. Dimensionless mechanical Principal Stress

The material properties of the surface layer are the same as zirconium except Young's modulus is 5 times that of zirconium. The material of the substrate is stellite III. $D = H/\ell = 2$, $\ell = 0.01$ in.

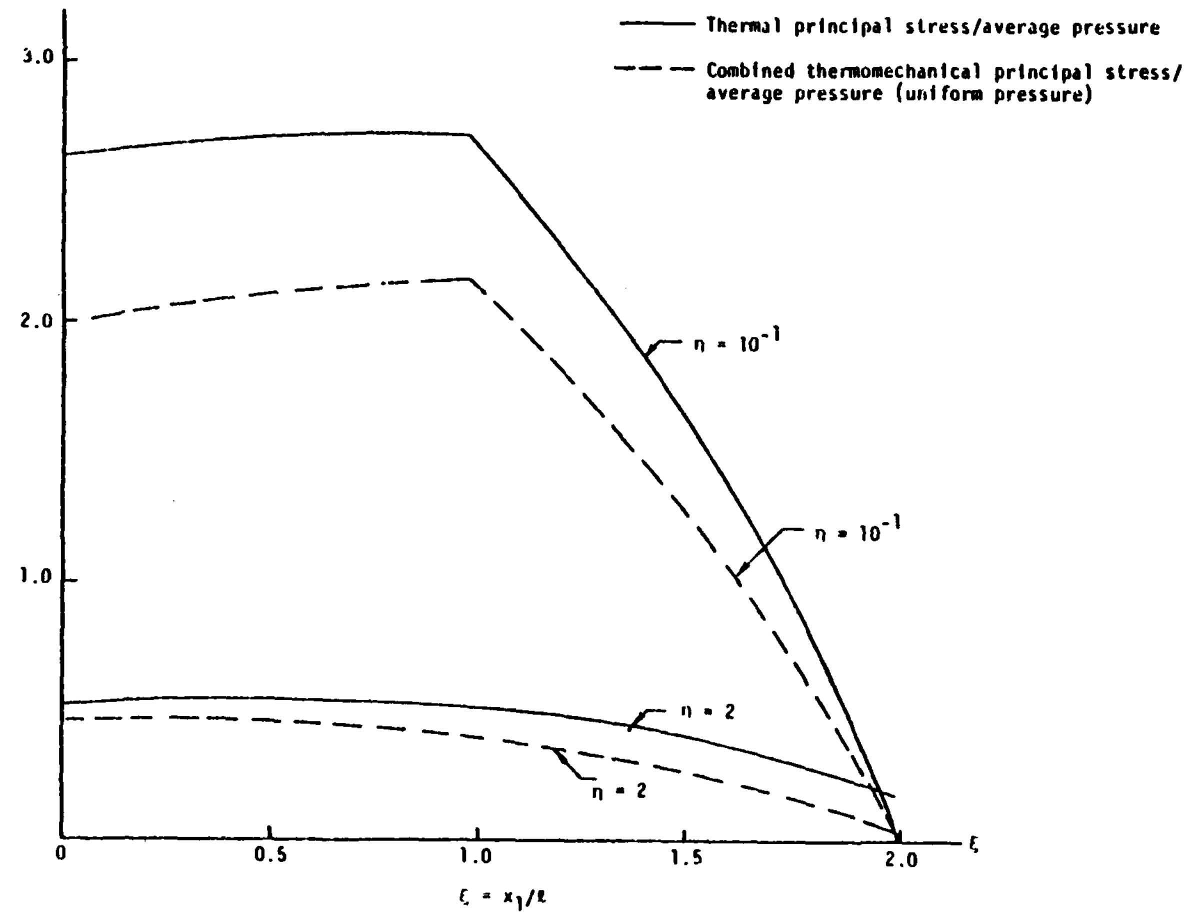

Figure 10. Dimensionless Thermal and Combined Thermomechanical Principal Stresses

The materials of the surface layer and the substrate are the same as Figure 9. $D = H/\ell = 2$, $\ell = 0.01$ in.

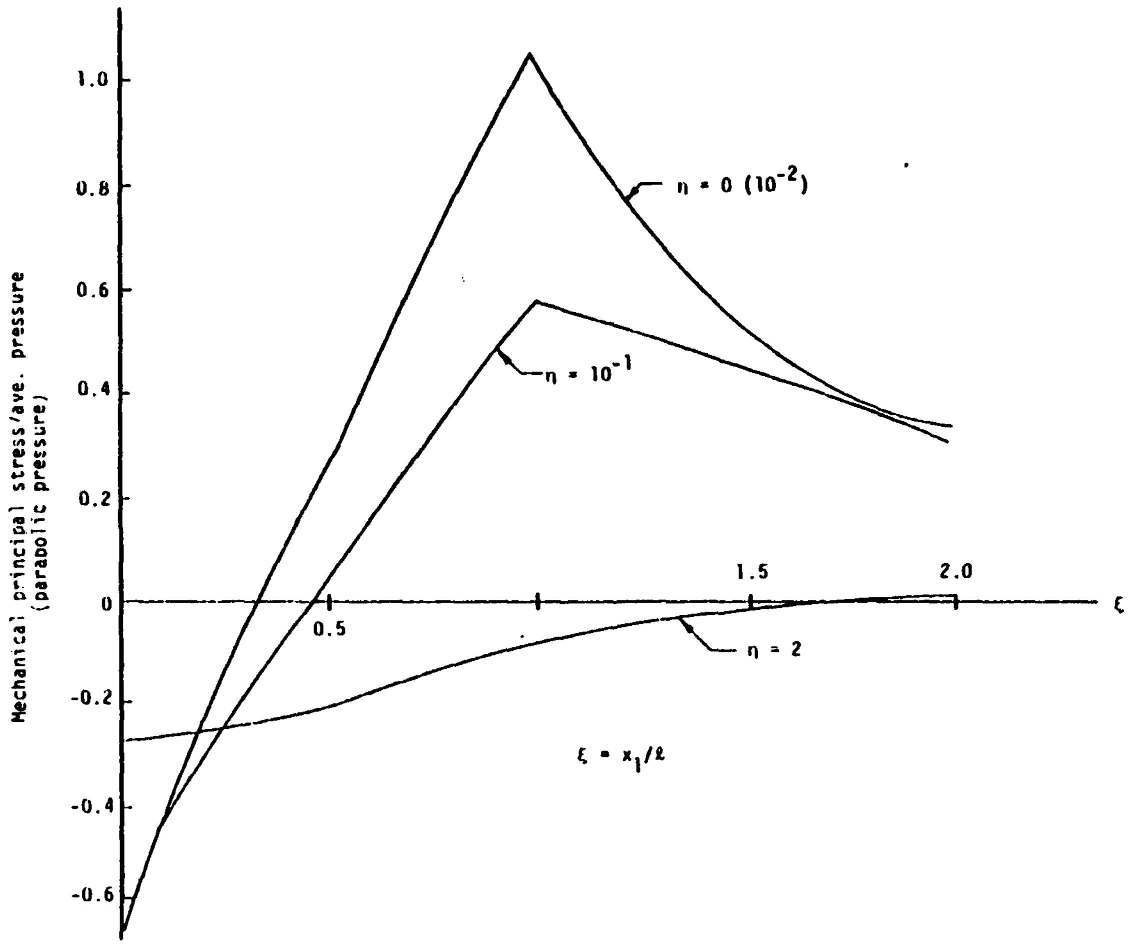

Figure 11. Dimensionless Mechanical Principal Stress

The material of the surface layer is zirconium. The material of the substrate is stellite III. $D = H/\ell = 2$, $\ell = 0.01$ in.

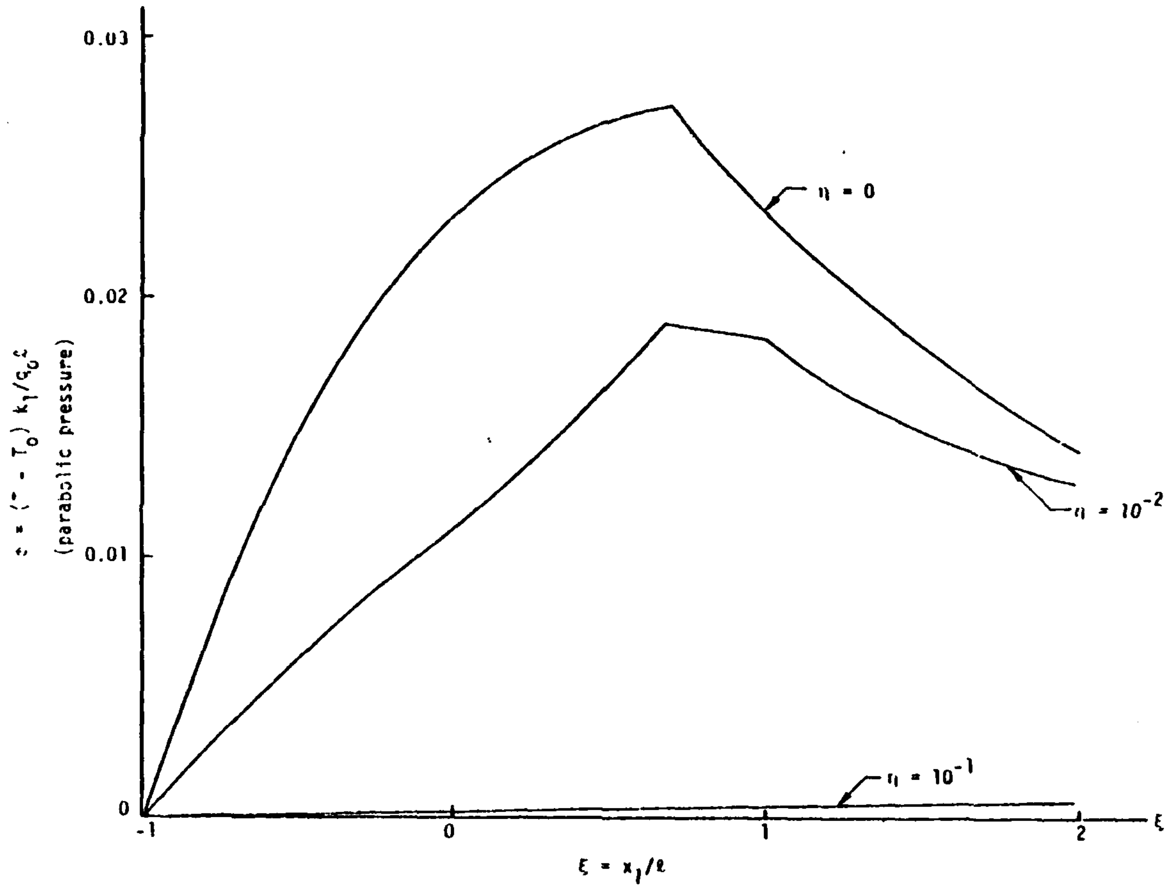

Figure 12. Dimensionless Temperature

The materials of the surface layer and the substrate are the same as Figure 1. $D = H/\ell = 2$, $\ell = 0.01$ in.

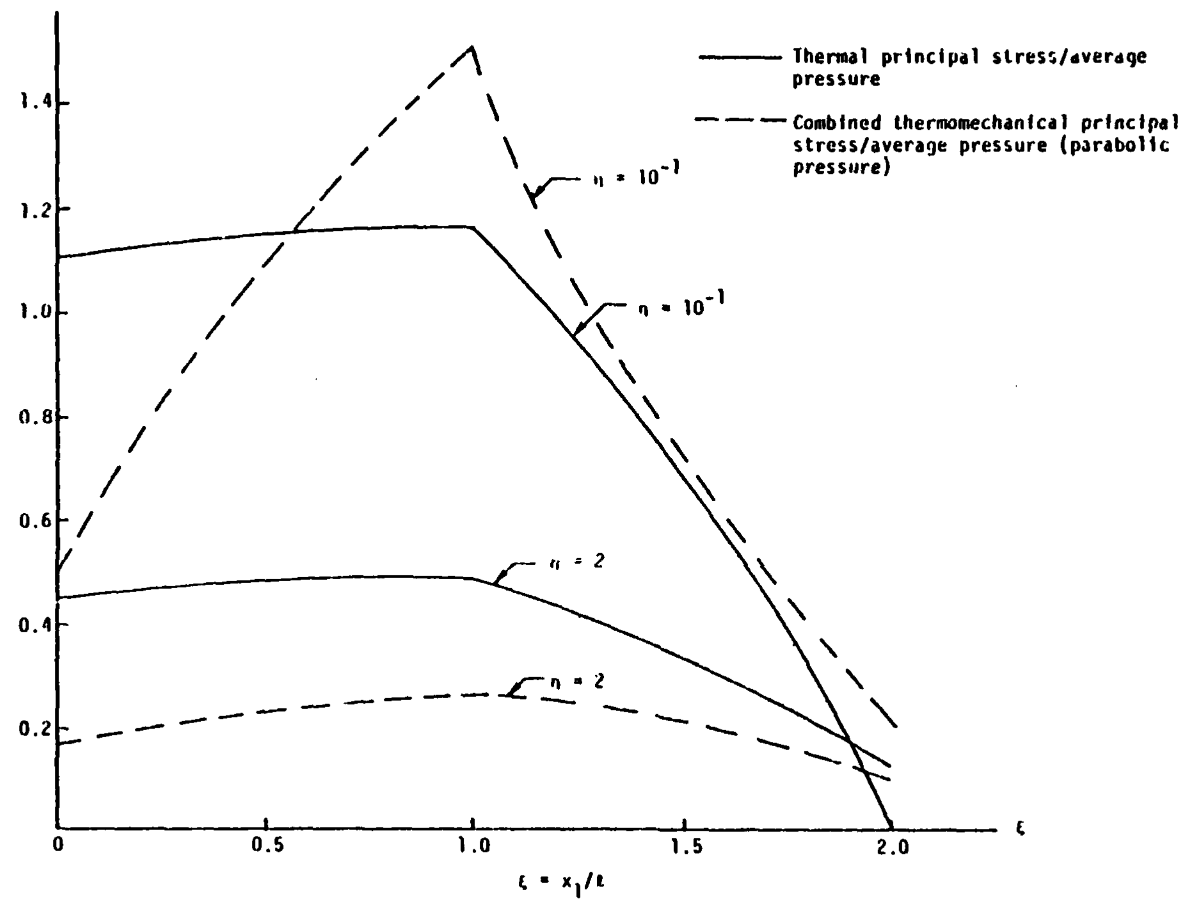

Figure 13. Dimensionless Thermal and Combined Thermomechanical Principal Stresses.

The materials of the surface layer and the substrate are the same as Figure 1. $D = H/\ell = 2$, $\ell = 0.01$ in.

APPENDIX I

THE EXPRESSIONS OF Δ, Δ_1, Δ_2, Δ_3, Δ_4, Δ_5, AND Δ_6

$$\Delta = \begin{vmatrix} 2 & 0 & (1+k^2) & 0 & 0 & 0 \\ 0 & j^{-1}(1+k^2) & 0 & 2k & 0 & 0 \\ j^{-1}s_j & j^{-1}c_j & ks_k & kc_k & f^{-1}e^{-fsD} & ge^{-gsD} \\ c_j & s_j & c_k & s_k & -e^{-fsD} & -e^{-gsD} \\ 2c_j & 2s_j & (1+k^2)c_k & (1+k^2)s_k & \frac{-2e^{-fsD}}{\delta J^2} & -\frac{(1+g^2)e^{-gsD}}{\delta J^2} \\ j^{-1}(1+k^2)s_j & j^{-1}(1+k^2)c_j & 2ks_k & 2kc_k & \frac{(1+g^2)e^{-fsD}}{f\delta J^2} & \frac{2g\ e^{-gsD}}{\delta J^2} \end{vmatrix}$$

$$= 2A_1 + (1 + k^2)A_2 + [2A_3 + (1 + k^2)A_4]\ c_jc_k + [2A_5 + (1 + k^2)A_6]c_js_k$$

$$+ [2A_7 + (1 + k^2)A_8]s_js_k + [2A_9 + (1 + k^2)A_{10}]c_ks_j$$

$$A_1 = \frac{k(1 + k^2)}{j}\left[\frac{4g}{(\delta J^2)^2} + \frac{2(1 + g^2)}{f\delta J^2} - \frac{4g}{\delta J^2} - \frac{(1 + g^2)^2}{f(\delta J^2)^2} - \frac{2g(1 + k^2)}{\delta J^2}\right.$$

$$\left. - \frac{2(1 + k^2)}{f} + 2g(1 + k^2) + \frac{(1 + k^2)(1 + g^2)}{f\delta J^2}\right]$$

$$A_2 = \frac{2k}{j}\left[\frac{4g}{(\delta J^2)^2} + \frac{(1 + k^2)(1 + g^2)}{f\delta J^2} - \frac{2g(1 + k^2)}{\delta J^2} - \frac{(1 + g^2)^2}{f(\delta J^2)^2} - \frac{4g}{\delta J^2}\right.$$

$$\left. - \frac{2(1 + K^2)}{f} + 2g(1 + k^2) + \frac{2(1 + g^2)}{f\delta J^2}\right]$$

$$A_3 = \frac{2k}{j}\left[-\frac{4g}{(\delta J^2)^2} - \frac{2(1+k^2)(1+g^2)}{f\delta J^2} + \frac{(1+g^2)^2}{f(\delta J^2)^2} + \frac{4g(1+k^2)}{\delta J^2}\right.$$

$$\left. + \frac{(1+k^2)^2}{f} - 8(1+k^2)^2\right]$$

$$A_4 = -\frac{k(1+k^2)}{j}\left[\frac{4g}{(\delta J^2)^2} + \frac{4(1+g^2)}{f\delta J^2} - \frac{8g}{\delta J^2} - \frac{(1+g^2)^2}{f(\delta J^2)^2} - \frac{4}{f} + 4g\right]$$

$$A_5 = \frac{2k^2}{j\delta J^2}\left[2(1+g^2) - 4 - (1+k^2)(1+g^2) + 2(1+k^2)\right]$$

$$A_6 = -\frac{g(1+k^2)}{jf\delta J^2}\left[-(1+k^2)(1+g^2) + 2(1+k^2) + 2(1+g^2) - 4\right]$$

$$A_7 = 2k^2\left[\frac{4g}{(\delta J^2)^2} + \frac{4(1+g^2)}{f\delta J^2} - \frac{8g}{\delta J^2} - \frac{(1+g^2)^2}{f(\delta J^2)^2} - \frac{4}{f} + 4g\right]$$

$$A_8 = -\frac{(1+k^2)}{j^2}\left[\frac{-4g}{(\delta J^2)^2} - \frac{2(1+k^2)(1+g^2)}{f\delta J^2} + \frac{(1+g^2)^2}{f(\delta J^2)^2} + \frac{4g(1+k^2)}{\delta J^2}\right.$$

$$\left. + \frac{(1+k^2)^2}{f} - g(1+k^2)^2\right]$$

$$A_9 = \frac{2kg}{f\delta J^2}\left[-(1+k^2)(1+g^2) + 2(1+k^2) + 2(1+g^2) - 4\right]$$

$$A_{10} = -\frac{k(1+k^2)}{j^2\ \delta J^2}\left[2(1+g^2) - 4 - (1+k^2)(1+g^2) + 2(1+k^2)\right]$$

$$
\Delta_1 = \begin{vmatrix}
\frac{i\mu_f \bar{P}}{\delta J^2} & 0 & (1+k^2) & 0 & 0 & 0 \\
-\frac{\bar{P}}{\delta J^2} & j^{-1}(1+k^2) & 0 & 2k & 0 & 0 \\
0 & j^{-1}c_j & ks_k & kc_k & f^{-1}e^{-fsD} & ge^{-gsD} \\
0 & s_j & c_k & s_k & -e^{-fsD} & -e^{-gsD} \\
0 & 2s_j & (1+k^2)c_k & (1+k^2)s_k & -\frac{2e^{-fsD}}{\delta J^2} & -\frac{(1+g^2)e^{-gsD}}{\delta J^2} \\
0 & j^{-1}(1+k^2)c_j & 2ks_k & 2kc_k & \frac{(1+g^2)e^{-fsD}}{f\delta J^2} & \frac{2ge^{-gsD}}{\delta J^2}
\end{vmatrix}
$$

$$
= e^{-(f+g)sD}\left[\frac{i\mu_f \bar{P}}{\delta J^2}(B_1 + B_2 c_j c_k + B_3 c_j s_k + B_4 s_j s_k + B_5 c_k s_j) - \frac{\bar{P}}{\delta J^2}(B_6 c_j s_k + B_7 c_j c_k + B_8 c_k s_j + B_9 s_j s_k)\right]
$$

$$
B_1 = \frac{k(1+k^2)}{j}\left[\frac{4g}{(\delta J^2)^2} + \frac{2(1+g^2)}{f\delta J^2} - \frac{4g}{\delta J^2} - \frac{(1+g^2)^2}{f(\delta J^2)^2} - \frac{2g(1+k^2)}{\delta J^2} - \frac{2(1+k^2)}{f} + 2g(1+k^2) + \frac{(1+k^2)(1+g^2)}{f\delta J^2}\right]
$$

$$
B_2 = \frac{2k}{j}\left[\frac{-4g}{(\delta J^2)^2} - \frac{2(1+k^2)(1+g^2)}{f\delta J^2} + \frac{(1+g^2)^2}{f(\delta J^2)^2} + \frac{4g(1+k^2)}{\delta J^2} + \frac{(1+k^2)^2}{f} - g(1+k^2)^2\right]
$$

$$B_3 = \frac{2k^2}{j\delta J^2} [2(1 + g^2) - 4 - (1 + k^2)(1+g^2) + 2(1 + k^2)]$$

$$B_4 = 2k^2[\frac{4g}{(\delta J^2)^2} + \frac{4(1 + g^2)}{f\delta J^2} - \frac{8g}{\delta J^2} - \frac{(1 + g^2)^2}{f(\delta J^2)^2} - \frac{4}{f} + 4g]$$

$$B_5 = \frac{2kg}{f\delta J^2} [- (1 + k^2)(1 + g^2) + 2(1 + k^2) + 2(1 + g^2) - 4]$$

$$B_6 = \frac{(1+k^2)}{j}\left[- \frac{4g}{(\delta J^2)^2} - \frac{2(1 + k^2)(1 + g^2)}{f\delta J^2} + \frac{(1 + g^2)^2}{f(\delta J^2)^2} + \frac{4g(1 + k^2)}{\delta J^2}\right.$$

$$\left. + \frac{(1 + k^2)^2}{f} - g(1 + k^2)^2\right]$$

$$B_7 = \frac{k(1 + k^2)}{j\delta J^2} [2(1 + g^2) - 4 - (1 + k^2)(1 + g^2) + 2(1 + k^2)]$$

$$B_8 = k(1 + k^2)\left[\frac{4g}{(\delta J^2)^2} + \frac{4(1 + g^2)}{f\delta J^2} - \frac{8g}{\delta J^2} - \frac{(1 + g^2)^2}{f(\delta J^2)^2} - \frac{4}{f} + 4g\right]$$

$$B_9 = \frac{g(1 + k^2)}{f\delta J^2} [- (1 + k^2) (1 + g^2) + 2(1 + k^2) + 2(1 + g^2) - 4]$$

$$\Delta_2 = \begin{vmatrix} 2 & \frac{i\mu_f\bar{P}}{\delta J^2} & (1+k^2) & 0 & 0 & 0 \\ 0 & \frac{-\bar{P}}{\delta J^2} & 0 & 2k & 0 & 0 \\ j^{-1}s_j & 0 & ks_k & kc_k & f^{-1}e^{-fsD} & ge^{-gsD} \\ c_j & 0 & c_k & s_k & -e^{-fsD} & -e^{-gsD} \\ 2c_j & 0 & (1+k^2)c_k & (1+k^2)s_k & -\frac{2e^{-fsD}}{\delta J^2} & -\frac{(1+g^2)e^{-gsD}}{\delta J^2} \\ j^{-1}(1+k^2)s_j & 0 & 2ks_k & 2kc_k & \frac{(1+g^2)e^{-fsD}}{f\delta J^2} & \frac{2g\ e^{-gsD}}{\delta J^2} \end{vmatrix}$$

$$= e^{-(f+g)sD}\Bigg[-\frac{i\mu_f\bar{P}}{\delta J^2}(c_1 s_j c_k + c_2 s_j s_k + c_3 c_j s_k + c_4 c_j c_k)$$

$$-\frac{\bar{P}}{\delta J^2}(c_5 + c_6 s_j s_k + c_7 s_j c_k + c_8 c_j c_k + c_9 s_k c_j)\Bigg]$$

$C_1 = B_2$, $C_2 = B_3$, $C_3 = B_4$, $C_4 = B_5$, $C_5 = jA_2$, $C_6 = -B_6$, $C_7 = -B_7$

$C_8 = -B_8$, $C_9 = -B_9$

$$
\Delta_3 = \begin{vmatrix}
2 & 0 & \dfrac{i\mu_f\bar{P}}{\delta J^2} & 0 & 0 & 0 \\
0 & j^{-1}(1+k^2) & \dfrac{-\bar{P}}{\delta J^2} & 2k & 0 & 0 \\
j^{-1}s_j & j^{-1}c_j & 0 & kc_k & f^{-1}e^{-fsD} & ge^{-gsD} \\
c_j & s_j & 0 & s_k & -e^{-fsD} & -e^{-gsD} \\
2c_j & 2s_j & 0 & (1+k^2)s_k & \dfrac{-2e^{-fsD}}{\delta J^2} & \dfrac{-(1+g^2)e^{-gsD}}{\delta J^2} \\
j^{-1}(1+k^2)s_j & j^{-1}(1+k^2)c_j & 0 & 2kc_k & \dfrac{(1+g^2)e^{-fsD}}{f\delta J^2} & \dfrac{2ge^{-gsD}}{\delta J^2}
\end{vmatrix}
$$

$$
= e^{-(f+g)sD}\left[\frac{\bar{P}}{\delta J^2}(d_1 s_k c_j + d_2 c_j c_k + d_3 s_j c_k + d_4 s_j s_k)\right.
$$

$$
\left. + \frac{i\mu_f\bar{P}}{\delta J^2}(d_5 s_j s_k + d_6 s_j c_k + d_7 c_j c_k + d_8 s_k c_j + d_9)\right]
$$

$d_1 = A_3/k$, $d_2 = A_5/k$, $d_3 = A_7/k$, $d_4 = A_9/k$, $d_5 = A_8$, $d_6 = A_{10}$,

$d_7 = A_4$, $d_8 = A_6$, $d_9 = A_2$

$$\Delta_4 = \begin{vmatrix} 2 & 0 & (1+k^2) & \dfrac{i\mu_f\bar{P}}{\delta J^2} & 0 & 0 \\ 0 & j^{-1}(1+k^2) & 0 & \dfrac{-\bar{P}}{\delta J^2} & 0 & 0 \\ j^{-1}s_j & j^{-1}c_j & ks_k & 0 & f^{-1}e^{-fsD} & ge^{-gsD} \\ c_j & s_j & c_k & 0 & -e^{-fsD} & -e^{-gsD} \\ 2c_j & 2s_j & (1+k^2)c_k & 0 & \dfrac{-2e^{-fsD}}{\delta J^2} & \dfrac{-(1+g^2)e^{-gsD}}{\delta J^2} \\ j^{-1}(1+k^2)s_j & j^{-1}(1+k^2)c_j & 2ks_k & 0 & \dfrac{(1+g^2)e^{-fsD}}{f\delta J^2} & \dfrac{2ge^{-gsD}}{\delta J^2} \end{vmatrix}$$

$$= e^{-(f+g)sD}\left[\frac{i\mu_f\bar{P}}{\delta J^2}\left(P_1 s_j c_k + P_2 s_j s_k + P_3 s_k c_j + P_4 c_j c_k\right)\right.$$

$$\left. - \frac{\bar{P}}{\delta J^2}\left(P_5 c_j c_k + P_6 s_k c_j + P_7 s_j s_k + P_8 s_j c_k + P_9\right)\right]$$

$P_1 = -A_8$, $P_2 = -A_{10}$, $P_3 = -A_4$, $P_4 = -A_6$, $P_5 = A_3/k$, $P_6 = A_5/k$,

$P_7 = A_7/k$, $P_8 = A_9/k$, $P_9 = A_1/k$

$$\Delta_5 = \begin{vmatrix} 2 & 0 & (1+k^2) & 0 & \frac{i\mu_f\bar{P}}{\delta J^2} & 0 \\ 0 & j^{-1}(1+k^2) & 0 & 2k & \frac{-\bar{P}}{\delta J^2} & 0 \\ j^{-1}s_j & j^{-1}c_j & ks_k & kc_k & 0 & ge^{-gsD} \\ c_j & s_j & c_k & s_k & 0 & -e^{-gsD} \\ 2c_j & 2s_j & (1+k^2)c_k & (1+k^2)s_k & 0 & \frac{-(1+g^2)e^{-gsD}}{\delta J^2} \\ j^{-1}(1+k^2)s_j & j^{-1}(1+k^2)c_j & 2ks_k & 2kc_k & 0 & \frac{2ge^{-gsD}}{\delta J^2} \end{vmatrix}$$

$$= e^{-gsD}\left[\frac{i\mu_f\bar{P}}{\delta J^2}(H_1s_j + H_2c_j + H_3c_k + H_4s_k) + \frac{\bar{P}}{\delta J^2}(H_5c_j + H_6s_j + H_7s_k + H_8c_k)\right]$$

$$H_1 = -\frac{k(1+k^2)}{j^2}\left[-2(1+k^2) + \frac{2(1+g^2)}{\delta J^2} + (1+k^2)^2 - \frac{(1+k^2)(1+g^2)}{\delta J^2}\right]$$

$$H_2 = -\frac{kg(1+k^2)}{j}\left[-2(1+k^2) + \frac{2(1+k^2}{\delta J^2}) + 4 - \frac{4}{\delta J^2}\right]$$

$$H_3 = -2k\left[-\frac{2j^{-1}g(1+k^2)}{\delta J^2} + j^{-1}g(1+k^2)^2 + \frac{4j^{-1}g}{\delta J^2} - 2j^{-1}g(1+k^2)\right]$$

$$H_4 = -\frac{2k^2}{j}\left[\frac{(1+k^2)(1+g^2)}{\delta J^2} - \frac{2(1+g^2)}{\delta J^2} - 2(1+k^2) + 4\right]$$

$$H_5 = \frac{2k}{j}\left[-2(1+k^2) + \frac{2(1+g^2)}{\delta J^2} + (1+k^2)^2 - \frac{(1+k^2)(1+g^2)}{\delta J^2}\right]$$

$$H_6 = 2kg\left[- 2(1 + k^2) + \frac{2(1 + k^2)}{\delta J^2} + 4 - \frac{4}{\delta J^2}\right]$$

$$H_7 = \frac{g(1 + k^2)}{j}\left[- \frac{2(1 + k^2)}{\delta J^2} + (1 + k^2)^2 + \frac{4}{\delta J^2} - 2(1 + k^2)\right]$$

$$H_8 = \frac{k(1 + k^2)}{j}\left[\frac{(1 + k^2)(1 + g^2)}{\delta J^2} - \frac{2(1 + g^2)}{\delta J^2} - 2(1 + k^2) + 4\right]$$

$$\Delta_6 = \begin{vmatrix} 2 & 0 & (1+k^2) & 0 & 0 & \frac{i\mu_f\bar{P}}{\delta J^2} \\ 0 & j^{-1}(1+k^2) & 0 & 2k & 0 & \frac{-\bar{P}}{\delta J^2} \\ j^{-1}s_j & j^{-1}c_j & ks_k & kc_k & f^{-1}e^{-fsD} & 0 \\ c_j & s_j & c_k & s_k & -e^{-fsD} & 0 \\ 2c_j & 2s_j & (1+k^2)c_k & (1+k^2)s_k & \frac{-2e^{-fsD}}{\delta J^2} & 0 \\ j^{-1}(1+k^2)s_j & j^{-1}(1+k^2)c_j & 2ks_k & 2kc_k & \frac{(1+g^2)e^{-fsD}}{f\delta J^2} & 0 \end{vmatrix}$$

$$= e^{-fsD}\left[\frac{-i\mu_f\bar{P}}{\delta J^2}(L_1s_j + L_2c_j + L_3c_k + L_4s_k) - \frac{\bar{P}}{\delta J^2}(L_5c_j + L_6s_j + L_7s_k + L_8c_k)\right]$$

$$L_1 = \frac{-k(1 + k^2)}{j^2}\left[- 2(1 + k^2) + \frac{4}{\delta J^2} + (1 + k^2)^2 - \frac{2(1 + k^2)}{\delta J^2}\right]$$

$$L_2 = \frac{-k(1 + k^2)}{j}\left[- \frac{2(1 + k^2)}{f} + \frac{(1 + k^2)(1 + g^2)}{f\delta J^2} + \frac{4}{f} - \frac{2(1 + g^2)}{f\delta J^2}\right]$$

$$L_3 = -\frac{2k}{j}\left[-\frac{(1+k^2)(1+g^2)}{f\delta J^2} + \frac{(1+k^2)^2}{f} + \frac{2(1+g^2)}{f\delta J^2} - \frac{2(1+k^2)}{f}\right]$$

$$L_4 = -\frac{2k^2}{j}\left[\frac{2(1+k^2)}{\delta J^2} - \frac{4}{\delta J^2} - 2(1+k^2) + 4\right]$$

$$L_5 = \frac{2k}{j}\left[-2(1+k^2) + \frac{4}{\delta J^2} + (1+k^2)^2 - \frac{2(1+k^2)}{\delta J^2}\right]$$

$$L_6 = 2k\left[-\frac{2(1+k^2)}{f} + \frac{(1+k^2)(1+g^2)}{f\delta J^2} + \frac{4}{f} - \frac{2(1+g^2)}{f\delta J^2}\right]$$

$$L_7 = \frac{(1+k^2)}{j}\left[-\frac{(1+k^2)(1+g^2)}{f\delta J^2} + \frac{(1+k^2)^2}{f} + \frac{2(1+g^2)}{f\delta J^2} - \frac{2(1+k^2)}{f}\right]$$

$$L_8 = \frac{k(1+k^2)}{j}\left[\frac{2(1+k^2)}{\delta J^2} - \frac{4}{\delta J^2} - 2(1+k^2) + 4\right]$$

APPENDIX II

THE EXPRESSIONS OF Δ_1^*, Δ_2^*, Δ_3^*, Δ_4^*, Δ_5^*, AND Δ_6^*

$$\Delta_1^* = \begin{vmatrix} E_{19} & 0 & (1+k^2) & 0 & 0 & 0 \\ E_{20} & j^{-1}(1+k^2) & 0 & 2k & 0 & 0 \\ E_{21} & j^{-1}c_j & ks_k & kc_k & f^{-1}e^{-fsD} & ge^{-gsD} \\ E_{22} & s_j & c_k & s_k & -e^{-fsD} & -e^{-gsD} \\ E_{23} & 2s_j & (1+k^2)c_k & (1+k^2)s_k & \dfrac{-2e^{-fsD}}{\delta J^2} & \dfrac{-(1+g^2)e^{-gsD}}{\delta J^2} \\ E_{24} & j^{-1}(1+K^2)c_j & 2ks_k & 2kc_k & \dfrac{(1+g^2)e^{-fsD}}{f\delta J^2} & \dfrac{2ge^{-gsD}}{\delta J^2} \end{vmatrix}$$

$$= e^{-(f+g)sD}\,[E_{19}(B_1 + B_2c_jc_k + B_3s_kc_j + B_4s_js_k + B_5s_jc_k)$$

$$+ (1 + k^2)(B_6s_kc_j + B_7c_jc_k + B_8s_jc_k + B_9s_js_k + B_{10}s_k + B_{11}c_k$$

$$+ B_{12}s_j + B_{13}c_j)]$$

where

$B_1 = A_1$

$B_2 = A_3$

$B_3 = A_5$

$B_4 = A_7$

$B_5 = A_9$

$$B_6 = \frac{E_{20}}{j}\left[- \frac{4g}{(\delta J^2)^2} - \frac{2(1 + k^2)(1 + g^2)}{f\delta J^2} + \frac{(1 + g^2)^2}{f(\delta J^2)^2} + \frac{4g(1 + k^2)}{\delta J^2} \right.$$

$$\left. + \frac{(1 + k^2)^2}{f} - g(1 + k^2)^2 \right]$$

$$B_7 = \frac{E_{20}k}{j\delta J^2}\left[2(1 + g^2) - 4 - (1 + k^2)(1 + g^2) + 2(1 + k^2) \right]$$

$$B_8 = E_{20}k\left[\frac{4g}{(\delta J^2)^2} + \frac{4(1 + g^2)}{f\delta J^2} - \frac{8g}{\delta J^2} - \frac{(1 + g^2)^2}{f(\delta J^2)^2} - \frac{4}{f} + 4g \right]$$

$$B_9 = \frac{E_{20}g}{f\delta J^2}\left[-(1 + k^2)(1 + g^2) + 2(1 + k^2) + 2(1 + g^2) - 4 \right]$$

$$B_{10} = \frac{-(1 + k^2)}{j}\left\{ E_{21}\left[-\frac{4g}{(\delta J^2)^2} - \frac{(1 + k^2)(1 + g^2)}{f\delta J^2} + \frac{(1 + g^2)^2}{f(\delta J^2)^2} + \frac{2g(1 + k^2)}{\delta J^2} \right] \right.$$

$$+ E_{22}\left[- \frac{g(1 + k^2)(1 + g^2)}{f\delta J^2} + \frac{2g(1 + k^2)}{f\delta J^2} \right] + E_{23}\left[\frac{g(1 + g^2)}{f\delta J^2} - \frac{2g}{f\delta J^2} \right]$$

$$\left. + E_{24}\left[\frac{2g}{\delta J^2} + \frac{(1 + k^2)}{f} - g(1 + k^2) - \frac{(1 + g^2)}{f\delta J^2} \right] \right\}$$

$$B_{11} = - \frac{(1 + k^2)}{j}\left\{ E_{21}\left[\frac{2k(1 + g^2)}{\delta J^2} - \frac{4k}{\delta J^2} \right] + E_{22}\left[\frac{4kg}{(\delta J^2)^2} + \frac{2k(1 + g^2)}{f\delta J^2} \right. \right.$$

$$\left. - \frac{4kg}{\delta J^2} - \frac{k(1 + g^2)^2}{f(\delta J^2)^2} \right] + E_{23}\left[- \frac{2kg}{\delta J^2} - \frac{2k}{f} + 2kg + \frac{k(1 + g^2)}{f\delta J^2} \right]$$

$$\left. + E_{24}\left[- \frac{k(1 + g^2)}{\delta J^2} + \frac{2k}{\delta J^2} \right] \right\}$$

$$
\begin{aligned}
B_{12} = 2k\Bigg\{ & E_{21}\left[-\frac{4g}{(\delta J^2)^2} - \frac{2(1+g^2)}{f\delta J^2} + \frac{(1+g^2)^2}{f(\delta J^2)^2} + \frac{4g}{\delta J^2} \right] \\
& + E_{22}\left[-\frac{2g(1+g^2)}{f\delta J^2} + \frac{4g}{f\delta J^2} \right] + E_{23}\left[\frac{g(1+g^2)}{f\delta J^2} - \frac{2g}{f\delta J^2} \right] \\
& + E_{24}\left[\frac{2g}{\delta J^2} + \frac{2}{f} - 2g - \frac{(1+g^2)}{f\delta J^2} \right] \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
B_{13} = 2k\Bigg\{ & E_{21}\left[\frac{(1+g^2)(1+k^2)}{j\delta J^2} - \frac{2(1+k^2)}{j\delta J^2} \right] + E_{22}\left[\frac{4g}{j(\delta J^2)^2} \right. \\
& \left. + \frac{(1+k^2)(1+g^2)}{jf\delta J^2} - \frac{2g(1+k^2)}{j\delta J^2} - \frac{(1+g^2)^2}{jf(\delta J^2)^2} \right] \\
& + E_{23}\left[-\frac{2g}{j\delta J^2} - \frac{(1+k^2)}{jf} + \frac{g(1+k^2)}{j} + \frac{(1+g^2)}{j\delta J^2} \right] \\
& + E_{24}\left[-\frac{(1+g^2)}{j\delta J^2} + \frac{2}{j\delta J^2} \right] \Bigg\}
\end{aligned}
$$

$$\Delta_2^* = \begin{vmatrix} 2 & E_{19} & (1+k^2) & 0 & 0 & 0 \\ 0 & E_{20} & 0 & 2k & 0 & 0 \\ j^{-1}s_j & E_{21} & ks_k & kc_k & f^{-1}e^{-fsD} & ge^{-gsD} \\ c_j & E_{22} & c_k & s_k & -e^{-fSD} & -e^{-gsD} \\ 2c_j & E_{23} & (1+k^2)c_k & (1+k^2)s_k & \dfrac{-2e^{-fsD}}{\delta J^2} & \dfrac{-(1+g^2)e^{-gsD}}{\delta J^2} \\ j^{-1}(1+k^2)s_j & E_{24} & 2ks_k & 2kc_k & \dfrac{(1+g^2)e^{-fsD}}{f\delta J^2} & \dfrac{2ge^{-gsD}}{\delta J^2} \end{vmatrix}$$

$$= e^{-(f+g)sD}\Big[2(c_1 + c_2c_k + c_3s_k) - E_{19}(c_4s_jc_k + c_5s_js_k + c_6c_js_k$$

$$+ c_7c_jc_k) + (1 + k^2)(c_8s_js_k + c_9s_jc_k + c_{10}c_jc_k + c_{11}s_kc_j$$

$$+ c_{12}c_j + c_{13}s_j)\Big]$$

where

$$c_1 = E_{20}k\left[\frac{4g}{(\delta J^2)^2} + \frac{(1 + k^2)(1 + g^2)}{f\delta J^2} - \frac{(1 + g^2)^2}{f(\delta J^2)^2} - \frac{2g(1 + k^2)}{\delta J^2} - \frac{4g}{\delta J^2}\right.$$

$$\left.- \frac{2(1 + k^2)}{f} + 2g(1 + k^2) + \frac{2(1 + g^2)}{f\delta J^2}\right]$$

$$c_2 = -\frac{2kj}{(1 + k^2)} B_{10}$$

$$c_3 = \frac{-2kj}{(1 + k^2)} B_{11}$$

$$c_4 = A_3$$

$$c_5 = A_5$$

$$c_6 = A_7$$

$$c_7 = A_9$$

$$c_8 = \frac{E_{20}A_8 j}{(1 + k^2)}$$

$$c_9 = \frac{E_{20}A_{10} j}{(1 + k^2)}$$

$$c_{10} = \frac{E_{20}A_4 j}{(1 + k^2)}$$

$$c_{11} = \frac{E_{20}A_6 j}{(1 + k^2)}$$

$$c_{12} = - B_{12}$$

$$c_{13} = - B_{13}$$

$$\Delta_3^* = \begin{vmatrix} 2 & 0 & E_{19} & 0 & 0 & 0 \\ 0 & j^{-1}(1+k^2) & E_{20} & 2k & 0 & 0 \\ j^{-1}s_j & j^{-1}c_j & E_{21} & kc_k & f^{-1}e^{-fsD} & ge^{-gsD} \\ c_j & s_j & E_{22} & s_k & -e^{-fsD} & -e^{-gsD} \\ 2c_j & 2s_j & E_{23} & (1+k^2)s_k & \frac{-2e^{-fsD}}{\delta J^2} & \frac{-(1+g^2)e^{-gsD}}{\delta J^2} \\ j^{-1}(1+k^2)s_j & j^{-1}(1+k^2)c_j & E_{24} & 2kc_k & \frac{(1+g^2)e^{-fsD}}{f\delta J^2} & \frac{2ge^{-gsD}}{\delta J^2} \end{vmatrix}$$

$$= e^{-(f+g)sD}\,[2(d_1 s_k + d_2 c_k + d_3 s_k c_j + d_4 c_j c_k + d_5 s_j c_k + d_6 s_j s_k + d_7 s_j + d_8 c_j) + E_{19}(d_9 s_j s_k + d_{10} s_j c_k + d_{11} c_j c_k + d_{12} s_k c_j + d_{13})]$$

where

$$d_1 = -B_{10}$$

$$d_2 = -B_{11}$$

$$d_3 = \frac{-E_{20}A_3}{2k}$$

$$d_4 = \frac{-E_{20}A_5}{2k}$$

$$d_5 = \frac{-E_{20}A_7}{2k}$$

$$d_6 = \frac{-E_{20}A_9}{2k}$$

$$d_7 = -B_{12}$$

$$d_8 = -B_{13}$$

$$d_9 = A_8$$

$$d_{10} = A_{10}$$

$$d_{11} = A_4$$

$$d_{12} = A_6$$

$$d_{13} = A_2$$

$$\Delta_4^* = \begin{vmatrix} 2 & 0 & (1+k^2) & E_{19} & 0 & 0 \\ 0 & j^{-1}(1+k^2) & 0 & E_{20} & 0 & 0 \\ j^{-1}s_j & j^{-1}c_j & ks_k & E_{21} & f^{-1}e^{-fsD} & ge^{-gsD} \\ c_j & s_j & c_k & E_{22} & -e^{-fsD} & -e^{-gsD} \\ 2c_j & 2s_j & (1+k^2)c_k & E_{23} & \dfrac{-2e^{-fsD}}{\delta J^2} & \dfrac{-(1+g^2)e^{-gsD}}{\delta J^2} \\ j^{-1}(1+k^2)s_j & j^{-1}(1+k^2)c_j & 2ks_k & E_{24} & \dfrac{(1+g^2)e^{-fsD}}{f\delta J^2} & \dfrac{2ge^{-gsD}}{\delta J^2} \end{vmatrix}$$

$$= e^{-(f+g)sD}\,[2(P_1c_k + P_2s_k + P_3c_jc_k + P_4s_kc_j + P_5s_js_k + P_6s_jc_k)$$

$$+ (1 + k^2)(P_7c_j + P_8s_j + P_9) - E_{19}(P_{10}s_jc_k + P_{11}s_js_k + P_{12}s_kc_j$$

$$+ P_{13}c_jc_k)]$$

where

$$P_1 = B_{10}$$

$$P_2 = B_{11}$$

$$P_3 = \frac{E_{20}A_3}{2k}$$

$$P_4 = \frac{E_{20}A_5}{2k}$$

$$P_5 = \frac{E_{20}A_7}{2k}$$

$$P_6 = \frac{E_{20}A_9}{2k}$$

$$P_7 = \frac{B_{12}(1 + k^2)}{2kj}$$

$$P_8 = \frac{B_{13}(1 + k^2)}{2kj}$$

$$P_9 = \frac{E_{20}A_1}{k(1 + k^2)}$$

$$P_{10} = A_8$$

$$P_{11} = A_{10}$$

$$P_{12} = A_4$$

$$P_{13} = A_6$$

$$\Delta_5^* = \begin{vmatrix} 2 & 0 & (1+k^2) & 0 & E_{19} & 0 \\ 0 & j^{-1}(1+k^2) & 0 & 2k & E_{20} & 0 \\ j^{-1}s_j & j^{-1}c_j & ks_k & kc_k & E_{21} & ge^{-gsD} \\ c_j & s_j & c_k & s_k & E_{22} & -e^{-gsD} \\ 2c_j & 2s_j & (1+k^2)c_k & (1+k^2)s_k & E_{23} & \frac{-(1+g^2)e^{-gsD}}{\delta J^2} \\ j^{-1}(1+k^2)s_j & j^{-1}(1+k^2)c_j & 2ks_k & 2kc_k & E_{24} & \frac{2ge^{-gsD}}{\delta J^2} \end{vmatrix}$$

$$= e^{-gsD}\,[2(H_1 + H_2s_jc_k + H_3s_js_k + H_4c_jc_k + H_5s_kc_j + H_6c_j$$

$$+ H_7s_j) + (1 + k^2)(H_8s_kc_j + H_9c_jc_k + H_{10}s_js_k + H_{11}s_jc_k + H_{12}$$

$$+ H_{13}s_k + H_{14}c_k) + E_{19}(H_{15}s_j + H_{16}c_j + H_{17}\,c_k + H_{18}s_k)]$$

where

$$H_1 = \frac{k(1+k^2)}{j}\left\{E_{21}\left[-2(1+k^2) + \frac{2(1+g^2)}{\delta J^2}\right] + E_{22}\left[-2g(1+k^2)\right.\right.$$

$$\left.\left. + \frac{2g(1+k^2)}{\delta J^2}\right] + E_{23}\left[2g - \frac{2g}{\delta J^2}\right] + E_{24}\left[(1+k^2) - \frac{(1+g^2)}{\delta J^2}\right]\right\}$$

$$H_2 = 2kg\left\{E_{21}\left[\frac{2(1+k^2)}{\delta J^2} - \frac{4}{\delta J^2}\right] + E_{24}\left[-(1+k^2) + 2\right]\right\}$$

$$H_3 = 2k^2\left\{E_{21}\left[-4 + \frac{2(1+g^2)}{\delta J^2}\right] + E_{22}\left[-4g + \frac{4g}{\delta J^2}\right] + E_{23}\left[2g - \frac{2g}{\delta J^2}\right]\right.$$

$$\left. + E_{24}\left[2 - \frac{(1+g^2)}{\delta J^2}\right]\right\}$$

$$H_4 = \frac{2k}{j}\left\{E_{21}\left[-\frac{(1+k^2)(1+g^2)}{\delta J^2} + (1+k^2)^2\right] + E_{22}\left[-\frac{2g(1+k^2)}{\delta J^2}\right.\right.$$

$$\left.\left. + g(1+k^2)^2\right] + E_{23}\left[\frac{2g}{\delta J^2} - g(1+k^2)\right] + E_{24}\left[\frac{(1+g^2)}{\delta J^2} - (1+k^2)\right]\right\}$$

$$H_5 = \frac{2k^2}{j}\left\{E_{22}\left[\frac{(1+k^2)(1+g^2)}{\delta J^2} - \frac{2(1+g^2)}{\delta J^2}\right] + E_{23}\left[-(1+k^2) + 2\right]\right\}$$

$$H_6 = E_{20}\,\frac{k}{j}\left[2(1+k^2) - \frac{2(1+g^2)}{\delta J^2} - (1+k^2)^2 + \frac{(1+k^2)(1+g^2)}{\delta J^2}\right]$$

$$H_7 = E_{20}(2kg)\left[(1+k^2) - \frac{(1+k^2)}{\delta J^2} - 2 + \frac{2}{\delta J^2}\right]$$

$$H_8 = -\frac{g(1+k^2)}{j}\left\{E_{21}\left[\frac{2(1+k^2)}{\delta J^2} - \frac{4}{\delta J^2}\right] + E_{24}\left[-(1+k^2) + 2\right]\right\}$$

$$H_9 = \frac{-H_5(1+k^2)}{2kj}$$

$$H_{10} = \frac{-H_6(1+k^2)}{2kj}$$

$$H_{11} = \frac{-k(1+k^2)}{j^2}\left\{E_{22}\left[\frac{(1+k^2)(1+g^2)}{\delta J^2} - \frac{2(1+g^2)}{\delta J^2}\right]\right.$$

$$\left. + E_{23}\left[-(1+k^2)+2\right]\right\}$$

$$H_{12} = \frac{2k}{j}\left\{E_{21}\left[-2(1+k^2) + \frac{(1+k^2)(1+g^2)}{\delta J^2}\right]\right.$$

$$+ E_{22}\left[-2g(1+k^2) + \frac{4g}{\delta J^2}\right] + E_{23}\left[g(1+k^2) - \frac{2g}{\delta J^2}\right]$$

$$\left. + E_{24}\left[2 - \frac{(1+g^2)}{\delta J^2}\right]\right\}$$

$$H_{13} = \frac{E_{20}g}{j}\left[\frac{2(1+k^2)}{\delta J^2} - (1+k^2)^2 - \frac{4}{\delta J^2} + 2(1+k^2)\right]$$

$$H_{14} = \frac{E_{20}k}{j}\left[-\frac{(1+k^2)(1+g^2)}{\delta J^2} + \frac{2(1+g^2)}{\delta J^2} + 2(1+k^2) - 4\right]$$

$$H_{15} = \frac{-k(1+k^2)}{j^2}\left[-2(1+k^2) + \frac{2(1+g^2)}{\delta J^2} + (1+k^2)^2\right.$$

$$\left. - \frac{(1+k^2)(1+g^2)}{\delta J^2}\right]$$

$$H_{16} = \frac{-kg(1+k^2)}{j}\left[-2(1+k^2) + \frac{2(1+k^2)}{\delta J^2} + 4 - \frac{4}{\delta J^2}\right]$$

$$H_{17} = \frac{-2kg}{j}\left[-\frac{2(1+k^2)}{\delta J^2} + (1+k^2)^2 + \frac{4}{\delta J^2} - 2(1+k^2)\right]$$

$$H_{18} = \frac{-2k^2}{j}\left[\frac{(1+k^2)(1+g^2)}{\delta J^2} - \frac{2(1+g^2)}{\delta J^2} - 2(1+k^2) + 4\right]$$

$$\Delta_6^* = \begin{vmatrix} 2 & 0 & (1+k^2) & 0 & 0 & E_{19} \\ 0 & j^{-1}(1+k^2) & 0 & 2k & 0 & E_{20} \\ j^{-1}s_j & j^{-1}c_j & ks_k & kc_k & f^{-1}e^{-fsD} & E_{21} \\ c_j & s_j & c_k & s_k & -e^{-fsD} & E_{22} \\ 2c_j & 2s_j & (1+k^2)c_k & (1+k^2)s_k & \dfrac{-2e^{-fsD}}{\delta J^2} & E_{23} \\ j^{-1}(1+k^2)s_j & j^{-1}(1+k^2)c_j & 2ks_k & 2kc_k & \dfrac{(1+g^2)e^{-fsD}}{f\delta J^2} & E_{24} \end{vmatrix}$$

$$= e^{-fsD}[2(L_1 + L_2s_jc_k + L_3s_js_k + L_4c_jc_k + L_5s_kc_j + L_6c_j + L_7s_j)$$

$$+ (1+k^2)(L_8s_kc_j + L_9c_jc_k + L_{10}s_js_k + L_{11}s_jc_k + L_{12} + L_{13}s_k$$

$$+ L_{14}c_k) - E_{19}(L_{15}s_j + L_{16}c_j + L_{17}c_k + L_{18}s_k)]$$

where

$$L_1 = \frac{k(1+k^2)}{j}\left\{E_{21}\left[2(1+k^2) - \frac{4}{\delta J^2}\right] + E_{22}\left[\frac{2(1+k^2)}{f} - \frac{(1+k^2)(1+g^2)}{f\delta J^2}\right]\right.$$

$$\left. + E_{23}\left[-\frac{2}{f} + \frac{(1+g^2)}{f\delta J^2}\right] + E_{24}\left[-(1+k^2) + \frac{2}{\delta J^2}\right]\right\}$$

$$L_2 = 2k\left\{E_{21}\left[-\frac{(1+k^2)(1+g^2)}{f\delta J^2} + \frac{2(1+g^2)}{f\delta J^2}\right] + E_{24}\left[\frac{(1+k^2)}{f} - \frac{2}{f}\right]\right\}$$

$$L_3 = 2k^2\left\{E_{21}\left[4 - \frac{4}{\delta J^2}\right] + E_{22}\left[\frac{4}{f} - \frac{2(1+g^2)}{f\delta J^2}\right] + E_{23}\left[-\frac{2}{f} + \frac{(1+g^2)}{f\delta J^2}\right]\right.$$

$$\left. + E_{24}\left[-2 + \frac{2}{\delta J^2}\right]\right\}$$

$$L_4 = \frac{2k}{j}\Bigg\{E_{21}\left[\frac{2(1+k^2)}{\delta J^2} - (1+k^2)^2\right] + E_{22}\left[\frac{(1+k^2)(1+g^2)}{f\delta J^2} - \frac{(1+k^2)^2}{f}\right]$$

$$+ E_{23}\left[-\frac{(1+g^2)}{f\delta J^2} + \frac{(1+k^2)}{f}\right] + E_{24}\left[-\frac{2}{\delta J^2} + (1+k^2)\right]\Bigg\}$$

$$L_5 = \frac{2k^2}{j}\Bigg\{E_{22}\left[-\frac{2(1+k^2)}{\delta J^2} + \frac{4}{\delta J^2}\right] + E_{23}\,[(1+k^2) - 2]\Bigg\}$$

$$L_6 = \frac{E_{20}k}{j}\left[-2(1+k^2) + \frac{4}{\delta J^2} + (1+k^2)^2 - \frac{2(1+k^2)}{\delta J^2}\right]$$

$$L_7 = \frac{E_{20}k}{f}\left[-2(1+k^2) + \frac{(1+k^2)(1+g^2)}{\delta J^2} + 4 - \frac{2(1+g^2)}{\delta J^2}\right]$$

$$L_8 = -\frac{(1+k^2)}{j}\Bigg\{E_{21}\left[-\frac{(1+k^2)(1+g^2)}{f\delta J^2} + \frac{2(1+g^2)}{f\delta J^2}\right] + E_{24}\left[\frac{(1+k^2)}{f} - \frac{2}{f}\right]\Bigg\}$$

$$L_9 = -\frac{(1+k^2)}{j}\Bigg\{E_{21}\left[4k - \frac{4k}{\delta J^2}\right] + E_{22}\left[\frac{4k}{f} - \frac{2k(1+g^2)}{f\delta J^2}\right]$$

$$+ E_{23}\left[-\frac{2k}{f} + \frac{k(1+g^2)}{f\delta J^2}\right] + E_{24}\left[-2k + \frac{2k}{\delta J^2}\right]\Bigg\}$$

$$L_{10} = -\frac{(1+k^2)}{j^2}\Bigg\{E_{21}\left[\frac{2(1+k^2)}{\delta J^2} - (1+k^2)^2\right] + E_{22}\left[\frac{(1+k^2)(1+g^2)}{f\delta J^2}\right.$$

$$\left.- \frac{(1+k^2)^2}{f}\right] + E_{23}\left[-\frac{(1+g^2)}{f\delta J^2} + \frac{(1+k^2)}{f}\right] + E_{24}\left[-\frac{2}{\delta J^2} + (1+k^2)\right]\Bigg\}$$

$$L_{11} = -\frac{(1+k^2)}{j^2}\Bigg\{E_{22}\left[-\frac{2k(1+k^2)}{\delta J^2} + \frac{4k}{\delta J^2}\right] + E_{23}\left[k(1+k^2) - 2k\right]\Bigg\}$$

$$L_{12} = \frac{2k}{j}\Bigg\{E_{21}\left[2(1+k^2) - \frac{2(1+k^2)}{\delta J^2}\right] + E_{22}\left[\frac{2(1+k^2)}{f} - \frac{2(1+g^2)}{\delta J^2}\right]$$

$$+ E_{23}\left[-\frac{(1+k^2)}{f} + \frac{(1+g^2)}{f\delta J^2}\right] + E_{24}\left[-2 + \frac{2}{\delta J^2}\right]\Bigg\}$$

$$L_{13} = \frac{E_{20}}{jf}\left[-\frac{(1+k^2)(1+g^2)}{\delta J^2} + (1+k^2)^2 + \frac{2(1+g^2)}{\delta J^2} - 2(1+k^2)\right]$$

$$L_{14} = \frac{E_{20}k}{j}\left[\frac{2(1 + k^2)}{\delta J^2} - \frac{4}{\delta J^2} - 2(1 + k^2) + 4\right]$$

$$L_{15} = -\frac{k(1 + k^2)}{j^2}\left[- 2(1 + k^2) + \frac{4}{\delta J^2} + (1 + k^2)^2 - \frac{2(1 + k^2)}{\delta J^2}\right]$$

$$L_{16} = -\frac{k(1 + k^2)}{j}\left[- \frac{2(1 + k^2)}{f} + \frac{(1 + k^2)(1 + g^2)}{f\delta J^2} + \frac{4}{f} - \frac{2(1 + g^2)}{f\delta J^2}\right]$$

$$L_{17} = \frac{-2k}{j}\left[- \frac{(1 + k^2)(1 + g^2)}{f\delta J^2} + \frac{(1 + k^2)^2}{f} + \frac{2(1 + g^2)}{f\delta J^2} - \frac{2(1 + k^2)}{f}\right]$$

$$L_{18} = -\frac{2k^2}{j}\left[\frac{2(1 + k^2)}{\delta J^2} - \frac{4}{\delta J^2} - 2(1 + k^2) + 4\right]$$

APPENDIX III

THE METHOD USED TO AVOID THE SINGULARITY IN THE INTEGRAL

In this appendix, the method used to avoid the numerical singularity of the integrals encountered in the analysis is discussed by means of the example integral $\int_0^4 \frac{1}{\sqrt{x}}\, dx$. This integral has a numerical singularity at $x = 0$, and therefore cannot be integrated numerically from zero to four. One approach to avoid the singularity is to replace the lower limit of the integral by ε which is an arbitrary small number greater than zero. Integration is carried out with decreasing values of ε until a value of the integral is obtained which is considerably different from the previous integral value corresponding to the previous value of ε. The last value of the integral thus obtained is taken to be the best approximation to the actual value of the integral. For the example

$$A = \int_0^4 \frac{1}{\sqrt{x}}\, dx$$

the exact solution is 4. The results of the numerical computation of A for various values of ε are tabulated below.

ε	A
10^{-9}	3.99994
10^{-10}	3.99998
10^{-11}	4.00000
10^{-12}	4.00005
10^{-13}	4.00018
10^{-14}	4.00063
10^{-15}	4.00204
10^{-16}	4.00647
10^{-17}	4.02051
10^{-18}	4.06490
10^{-19}	4.20526
10^{-20}	4.64914

The table indicates that the numerical integration diverges for values of ε smaller than 10^{-18}. $\varepsilon = 10^{-18}$ therefore gives the best numerical solution to the problem.

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